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On Piecewise Monotone Interval Maps and Periodic Points

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Abstract

In this short note, we find that a continuous piecewise monotone interval map f is chaotic in the sense of Li and Yorke if and only if f restricted to the set of its periodic points is not Lyapunov stable.

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1 Introduction

Let $f : I = [0,1] \to I$ be a continuous map. In the huge list of conditions which are equivalent to zero entropy of a continuous interval map (see e.g. [14]), there were three, (C2)–(C4), included for a long time:

(C1) The map f has zero topological entropy (see [1] for the definition).

(C2) The map $f|_{\mathrm{P}(f)}$ is Lyapunov stable (it has equicontinuous powers).

(C3) The set $\mathrm{R}(f)$ is a F_σ set.

(C4) The set $\mathrm{P}(f)$ is a G_δ set.

Recall that the orbit of a point $x \in I$ is given by the sequence $(f^n(x))$, where $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n > 1$. A point $x \in I$ is periodic provided there is $n \in \mathbb{N}$ such that $f^n(x) = x$. The smallest positive integer n holding this condition is called the period of x. The limit points of the orbit of x is called the ω–limit set of x under f, denoted by $\omega(x, f)$. A point x is called recurrent if $x \in \omega(x, f)$. Denote by $\mathrm{P}(f)$ and $\mathrm{R}(f)$ the sets of periodic and recurrent points of f, respectively. Recall that a subset A is a G_δ set provided it is equal to the intersection of a countable collection of open subsets. The set F is an F_σ if it is the countable union of closed sets.

At the beginning of XXI century, the equivalence among these properties were proved to be false. In [15] was proved that condition (C1) was not equivalent to (C2), although condition (C2) always implies (C1). A similar result was proved in [17] concerning conditions (C1) and (C3). Finally, in [16], the equivalence between (C1) and (C4) is disproved by proving that (C1) does not imply (C4), and in [11] has been recently proved that (C4) does not imply (C3).

If we think about these properties for a while, we see that conditions (C3) and (C4) are related to the topological structure of two sets from the topological dynamics of f. Property (C2) is a dynamical property itself, because states that the dynamics of $f|_{\mathrm{P}(f)}$ is quite simple. Let us point out that, recently, in [5] and [6] the dynamics of f has been studied from the set of periodic points of the map f.

The maps of the above mentioned counterexamples for (C3) and (C4) were obtained as functional limit of continuous maps and hence, they are not piecewise monotone. Recall that $f : I \to I$ is piecewise monotone if there is a partition $0 = x_0 < x_1 < ... < x_n = 1$ of I such that $f|_{(x_i, x_{i+1})}$ is monotone for $i = 0, ..., n-1$. This fact was not strange for conditions (C3) and (C4), because these equivalences are true for such kind of maps (see [17]). The counterexample on property (C2) was constructed by a so–called weakly unimodal map, which has two pieces of monotonicity. The aim of this paper is to go further and proving the following result.

Theorem 1 *Let $f : I \to I$ be a piecewise monotone continuous map. Then the map $f|_{\mathrm{P}(f)}$ is Lyapunov stable if and only if f is not chaotic in the sense of Li and Yorke.*

Recall that a continuous interval map f is chaotic in the sense of Li and Yorke (LY–chaotic) if there is an uncountable set $S \in I$ such that

$$0 = \liminf_{n\to\infty} |f^n(x) - f^n(y)| < \limsup_{n\to\infty} |f^n(x) - f^n(y)|,$$

for any $x, y \in S$, $x \neq y$. In addition, we say that f is LY–simple if for any $x \in I$ and any $\varepsilon > 0$, there is a periodic point y such that $\limsup_{n\to\infty} |f^n(x) - f^n(y)| < \varepsilon$.

Hence, we can add our main result to the following one. First, we recall that $\omega(f) = \cup_{x \in I} \omega(x, f)$. $\Omega(f)$ denotes the set of nonwandering points, that is, those points $x \in I$ such that for any $\varepsilon > 0$ there is $n > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap f^n(x - \varepsilon, x + \varepsilon) \neq \emptyset$. $\mathrm{AP}(f)$ is the set of almost periodic points for which for any $\varepsilon > 0$ there is $k > 0$ such that $f^{kn}(x) \in (x - \varepsilon, x + \varepsilon)$ for any $n \geq 0$. For the definition of topological sequence entropy, which is an extension of topological entropy, see [10] or [9].

Corollary 1 *Let $f : I \to I$ be a piecewise monotone continuous map. The following statements are equivalent:*

(a) *The map* f *is not LY–chaotic.*

(b) *The map* f *is LY–simple.*

(c) $\omega(f) = \{x \in [0,1] : \lim_{n\to\infty} f^{2^n}(x) = x\}$.

(d) $\mathrm{AP}(f) = \omega(f)$.

(e) *The map* $f|_{\omega(f)}$ *is Lyapunov stable.*

(f) *The map* $f|_{\Omega(f)}$ *is Lyapunov stable.*

(g) *The topological sequence entropy of* f*,* $h_A(f)$*, is zero for any increasing sequence of integers* A*.*

(h) *The map* $f|_{\mathrm{P}(f)}$ *is Lyapunov stable.*

We remark that the properties (b)–(g) in Corollary 1 are equivalent to (a) without the assumption that f is piecewise monotone as one can see in the references [7], [9] and [8]. The equivalence between properties (a) and (h) comes from Theorem 1. On the other hand, there is an example in [7] proving that condition (h) cannot imply (a) without the assumption that f is piecewise monotone.

We will prove our main result in next section.

2 Proof of Theorem 1

Before proving our main result, we will show the following one, whose proof is immediate.

Proposition 2 *Let* $f : [0,1] \to [0,1]$ *be non LY–chaotic. Then* $f|_{\mathrm{P}(f)}$ *is Lyapunov stable.*

Proof. Since f is not chaotic, by [7], we have that $f|_{\omega(f)}$ is Lyapunov stable. The result follows because $\mathrm{P}(f) \subset \omega(f)$.□

Proof of Theorem 1. In view of Proposition 2, we just need to prove that if f is LY–chaotic, then $f|_{\mathrm{P}(f)}$ cannot be Lyapunov stable. Recall that a LY–chaotic map with zero topological entropy has an infinite ω–limit set $\omega(x,f)$ with the following properties (see [18]):

- There is a nested sequence of intervals $J_0 \supset J_1 \supset ... \supset J_n \supset ...$ such that $f^{2^n}(J_n) = J_n$ and
$$\omega(x,f) \subset \bigcap_{n\geq 0} \bigcup_{j=0}^{2^n-1} f^j(J_n).$$

- $\omega(x,f)$ contains two f–nonseparable points u, v, that is, for any $n \geq 1$, u and v are contained in the same periodic interval $f^j(J_n)$.

Now, we consider the set $\Sigma = \{0,1\}^{\mathbb{N}}$, and for any $\alpha \in \Sigma$ and $n \in \mathbb{N}$, let $\alpha|_n = (\alpha_1, ..., \alpha_n)$. We write $J_{\mathbf{0}|_n} = J_n$, where $\mathbf{0} = (0,0,...)$. Then, denote by $J_{\alpha|_n} = f^j(J_n)$ in such a way that $\alpha|_n = a_n^j(\mathbf{0}|_n)$, where $a_n(1,1,...,1) = \mathbf{0}|_n$ and $a_n(\alpha|_n) = \alpha|_n * 1$ for $\alpha|_n \neq (1,...,1)$, where $*$ denotes the operation which adds 1 to α_1; if $\alpha_1 + 1 = 1$, then $a_n(\alpha|_n) = (1, \alpha_2, ..., \alpha_n)$, if $\alpha_1 + 1 = 2$, then we put 0 in the first component and add 1 to α_2 and repeat this porcces untill α_j will be 1. For instance $a_3(1,1,0) = (0,0,1)$ and $a_3(0,1,1) = (1,1,1)$. Clearly, for $\alpha \in \Sigma$ and $n < m$, we have that $J_{\alpha|_m} \subset J_{\alpha|_n}$. Denote by $J_\alpha = \cap_{n\geq 1} J_{\alpha|_n}$

For a subinterval J, $|J|$ will be its length. Now, let $\delta > 0$. Let $\mathcal{A}_\delta = \{\alpha \in \not< : |J_\alpha| \geq \delta\}$. Now, we claim that there exists an $n_\delta \in \mathbb{N}$ such that for any $n \geq n_\delta$ it is held

- if $\alpha \in \mathcal{A}_\delta$ then $\max\{|J^+_{\alpha|_n}|, |J^-_{\alpha|_n}|\} < \delta$, where $J^+_{\alpha|_n}$ and $J^-_{\alpha|_n}$ are the right and left side subintervals of $J_{\alpha|_n} \setminus J_\alpha$.
- if $\theta \in \{0,1\}^n$ and $\alpha|_n \neq \theta$ for any $\alpha \in \mathcal{A}_\delta$ then $|J_\theta| < \delta$.

To prove our claim, let $\alpha \in \mathcal{A}_\delta$. Since $(J_{\alpha|_n})_{n=1}^\infty$ decreases to J_α, if n is large enough then $\max\{|J^+_{\alpha|_n}|, |J^-_{\alpha|_n}|\} < \delta$. Since $\mathcal{A}_\delta$ is finite we have $\max\{|J^+_{\alpha|_n}|, |J^-_{\alpha|_n}|\} < \delta$ for all $\alpha \in \mathcal{A}_\delta$ and all sufficient large n. Now, we show that if n is large enough then $|J_\theta| < \delta$ for any $\theta \in \{0,1\}^n$ with the property $\alpha|_n \neq \theta$ for all $\alpha \in \mathcal{A}_\delta$. Suppose the contrary. Then there are a strictly increasing sequence $(n_j)_{j=1}^\infty$ and sequences $\theta^j \in \{0,1\}^{n_j}$ such that $|J_{\theta^j}| \geq \delta$ and $\alpha|_j \neq \theta^{n_j}$ for any $\alpha \in \mathcal{A}_\delta$. Let x_j be the midpoint of J_{θ^j}. It is clearly not restrictive to assume that $(x_j)_{j=1}^\infty$ converges to some x and $|x_j - x| < \delta/2$ for any j. Since for any fixed n all intervals J_θ, $\theta \in \{0,1\}^n$, are pairwise disjoint, this means that each pair K_{θ^j} and $K_{\theta^{j+1}}$ has non-empty intersection, which clearly implies $J_{\theta^{j+1}} \subset J_{\theta^j}$ for any j and hence the existence of an $\alpha \in \Sigma$ with $\alpha|_{n_j} = \theta^j$ for any j. Due to the definition of the intervals J_{θ^j}, α cannot belong to $\mathcal{A}_\delta$. However, $J_\alpha = \bigcap_{n=1}^\infty J_{\alpha|_n} = \bigcap_{j=1}^\infty J_{\theta^j}$ so $|J_\alpha| \geq \delta$, a contradiction.

Now, fix $\varepsilon = |u - v|$. Since $\overline{\mathrm{P}}(f) = \omega(f)$ (cf. [4]), there are sequences of periodic points u_n and v_n which converge to u and v, respectively. Now, fix $\delta > 0$, $\delta < \varepsilon$, and $\mathcal{A}_\delta$ as before. There is $n_0 \in \mathbb{N}$ such that u_n and v_n are contained in $J^+_{\alpha|_n} \cup J^-_{\alpha|_n}$, where $\alpha \in \Sigma$ is such that $u, v \in J_\alpha$. Since u_n and v_n are periodic points, there is $\theta \in \{0,1\}^n$, $n \geq \max\{n_\delta, n_0\}$ such that $f^j(u_n)$ and $f^j(v_n)$ belong to J_θ for some $0 < j < 2^n$ and such that $|J_\theta| < \delta$. Hence $|f^j(u_n) - f^j(v_n)| < \delta$ and $|f^{2^n - j}(f^j(u_n)) - f^{2^n-j}(f^j(v_n))| > \varepsilon$, which proves that $f|_{\mathrm{P}(f)}$ cannot be Lyapunov stable.□

Remark 3 *Recall that a wandering interval of f is an interval whose iterates are pairwise disjoint and such that neither of the orbits of its points is attracted by any periodic orbit. In fact if a map $f \in C(I)$ of zero entropy is chaotic then it must possess a wandering interval (see e.g. [2]). In many "natural" maps (including all analytic ones) wandering intervals cannot exist [12] and then they cannot be LY–chaotic. So, we wonder about the validity of Theorem 1 under regularity conditions of f, for instance for C^1 maps.*

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