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Symmetry and Symmetry-Breaking of the Emergent Dynamics of the Discrete Stochastic Majority-Voter Model

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Abstract

We analyse the emergent dynamics of the so called majority voter model evolving on complex networks. In particular we study the influence of three characteristic types of networks, namely Random Regular, Erdös-Rényi (ER), Watts and Strogatz (WS, small-world) and Barabasi (scale- free) on the bifurcating stationary coarse-grained solutions. We first prove analytically some simple properties about the symmetry and symmetry breaking of the macroscopic dynamics with respect to the network topology. We also show how one can exploit the Equation-free framework to bridge in a computational strict manner the micro to macro scales of the dynamics of stochastic individualistic models on complex random graphs. In particular, we show how systems-level tasks such as bifurcation analysis of the coarse-grained dynamics can be performed bypassing the need to extract macroscopic models in a closed form. A comparison with the mean-field approximations is also given illustrating the merits of the Equation-Free approach, especially in the case of scale-free networks exhibiting a heavy-tailed connectivity distribution.

Keywords: Majority-Voter Model, Micro to Macro bridging, Coarse-grained computations, Equation-free approach, Complex Networks, Nonlinear Dynamics.

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1 Introduction

Social-like mimetic behavior - otherwise described as trend following or majority-voter or majority-rule- has been, over the last century, a key factor in determining and shaping economical and political changes around the world [20, 45, 23]. Taking as an example decision making, it has been demonstrated that the mimetic behavior of individuals may significantly affect rational decisions under incomplete information [39, 37, 49, 16]. Identifying and understanding collective actions from such phenomena has therefore been an important research task for psychologists, sociologists and economists. Over the years, scientists have extensively used this mechanism to model and gain a better understanding on the behavior of opinion formation and voter/election dynamics [8, 19, 48, 7, 32, 33] culture and language dynamics [11, 3, 6, 10, 35], crowd flow design and management [18, 21, 17, 40], diffusion of news and innovations [15, 34, 51],but also epidemic spread dynamics [5, 4, 13, 28, 27, 42] ecology [24] and neuroscience [31, 46, 47]. Given the nature of the process, it is clear that the network topology, on which the interaction of the individuals evolves, can shape the emergent macroscopic dynamics. However, it is less clear how one can quantify in a systematic manner the dependence of the emergent dynamics with respect to both network characteristics and model parameters. Due to the nonlinear, stochastic nature of such models and their coupling to complex network structures, the emergent behavior cannot be-most of the times-accurately modeled and analyzed in a straightforward manner.

While one can try to use the tools of statistical physics to write down coarse-grained master equations to describe the probabilistic time evolution of the macroscopic quantities for simple-structured homogeneous networks (in the sense that there is a constant degree connectivity and/or that the structure is poorly clustered), major problems arise in trying to find fair- or perform computations based on- macroscopic models in a closed form when dealing with complex heterogeneous networks (such as scale-free type of networks) [50, 1, 38]. This imposes a major obstacle to systems-level computational tasks, such as bifurcation and stability analysis which rely on the availability of efficient low-order closed models written in terms of a few macroscopic (coarse-grained) variables. Hence, being able to systematically analyze the dynamics of majority-voter processes on complex networks becomes, in this context, of great importance.

Here, our main focus is to systematically explore the dynamics of the basic majority-voter process deploying on complex networks. We first prove analytically some symmetry properties of the corresponding mean field models. We then show how the network structure induces symmetry breaking of the system solutions giving rise to hysteresis phenomena. Asymmetric behavior predicted by detailed network models has been observed in many real-world complex problems. In particular, symmetry breaking of majority-voter processes has been related to phenomena such as herd behavior under panic [2], the emergence of cooperation dynamics [41] and public opinion formation [22]. Finally, in order to systematically analyse the way symmetry breaking influences the emergent dynamics we exploit the Equation-Free framework [29, 14, 44, 36] bypassing the construction of explicit coarse-grained models. In particular, we construct the coarse-grained bifurcation diagrams and perform a stability analysis of the basic majority-rule dynamics evolving on complex networks, with respect to (a) model's switching-state probability and (b) to the underlying degree distribution. We should note that this is the first time that such an analysis is provided using the detailed stochastic model in an explicit manner, i.e. bypassing the need to construct mean-field approximations. A comparison with the mean field approximations is also demonstrated, to reveal the merits of the proposed framework. In particular in the case of scale-free structures even if one manages to extract exact mean filed approximations, we show that bifurcation analysis, based on the corresponding analytical mean field approximation, appears to be an overwhelming difficult task, due to the heavy tail power-law connectivity distribution.

The paper is organized as follows. In section 2 we describe the majority-voter model deploying on a network while in section 3 we prove analytically how the connectivity degree of random graphs governs the symmetry and the symmetry breaking of the solutions of the corresponding mean field models. In section 4 we derive the mean field approximation of the majority-voter model in the case of complex networks with arbitrary degree distributions. In section 5, we show how the Equation-free framework can be exploited to perform systems level tasks on complex networks. In section 6 we present the results of the coarse-grained numerical analysis, constructing the coarse-grained bifurcation diagrams of the majority-voter dynamics as these obtained by exploiting the Equation-free approach on complex networks. A comparison with the bifurcation diagrams obtained using the mean field approximations is also made. We conclude in section 7.

2 The discrete stochastic majority-voter model

In our simple basic majority-voter model [31, 46, 47], each individual is labeled as i, (i=1,2,...,N), which votes for "A" or "B". Hence, the state of the *i-th* individual in time is described with the function $a_i(t) \in \{0, 1\}$, where the values 1 and 0 corresponds to "A" and "B" selection respectively. Let us denote by $\Lambda(i)$ the set of the neighbors (i.e. the individuals connected to *i-th* individual, with self loop included). The summation

$$\sigma_i(t) = \sum_{j \in \Lambda(i)} a_i(t) \tag{1}$$

gives the number of individuals socially linked with the *i-th* individual voting for "A". At each time step each individual is influenced by its social interactions, and changes its preference according to the following simple stochastic rules:

1. A "B" voter changes its preferences to "A" with probability ε, if $\sigma_i(t) \leq \left(\frac{k_i}{2}\right)$ (where k_i is the degree of the $i-th$ individual). If $\sigma_i(t) > \left(\frac{k_i}{2}\right)$ the individual keeps changes its preferences to "A" with probability $1 - \varepsilon$.

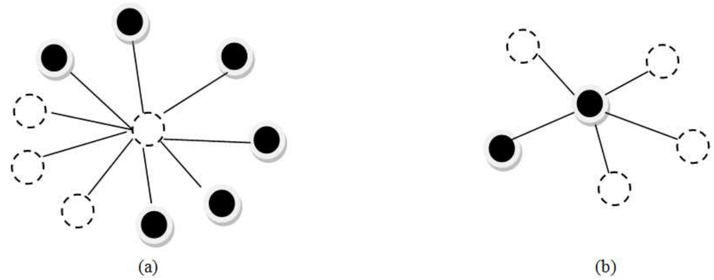

Figure 1: An example of the majority-voter rules. The open circle represents a "B" voter, while the filled circle represents an "A" voter (a). $p_{B\to A} = 1-\varepsilon$, since $\sigma(t) = \sum_{j=1}^{10} a_j(t) = 6 > degree(i)/2$. Here $degree$(i)=10 (self loop is included). (b) Similar, $p_{A\to B} = 1-\varepsilon$, since $\sigma(t) = \sum_{j=1}^{6} a_j(t) = 2 < degree(i)/2$.

2. An individual with a preference "A" changes its preference to "B" with probability ϵ, if $\sigma_i(t) > \left(\frac{k_i}{2}\right)$. If $\sigma_i(t) \leq \left(\frac{k_i}{2}\right)$, the individual changes its preference to B with probability $1-\varepsilon$. The probability ε ranges in $(0, 0.5)$. We illustrate the operation of the above rules in Fig. 1.

3 Symmetry and symmetry breaking of the solutions of the mean-field majority-voter model evolving on random regular networks (RRN)

In the following section we prove some simple but important properties of the majority-voter dynamics evolving on random regular networks.

3.1 RRN with an odd connectivity distribution

For our analysis we start by considering a random network with constant, odd, connectivity degree: $k = 2l-1$ and $l \in N$ (self loop is included). In this case, the time evolution of the density of A voters is given by the equation

$$d_{t+1} = f_{2l-1}(d_t) \tag{2}$$

where the function f_{2l-1} takes the form

$$\begin{aligned} f_{2l-1}(d,\varepsilon) = (1-\varepsilon)(\binom{2l-1}{0} d^{2l-1} + \binom{2l-1}{1} d^{2l}(1-d) + ... + \binom{2l-1}{l-1} d^{l}(1-d)^{l-1}) + \\ + \varepsilon(\binom{2l-1}{l} d^{l-1}(1-d)^{l} + \binom{2l-1}{l+1} d^{l-2}(1-d)^{l+1} + ... + \binom{2l-1}{2l-1}(1-d)^{2l-1}) \end{aligned} \tag{3}$$

In this case, the following proposition holds:

Proposition 3.1.1. *Let (G,E) be a network with constant odd connectivity $k = 2l-1$, $l \in N$. Then the fixed point solutions of equation (2) for a constant ε are symmetric with respect to $d = 1/2$.*

Proof. We shall prove that if d_0 is a solution of Eq. (2), then $1-d_0$ is also a solution. The function f_{2l-1}, Eq. (3) can be written in a more compact form

$$f_{2l-1}(d,\varepsilon) = (1-\varepsilon)\sum_{i=0}^{l-1}\binom{2l-1}{i}d^{2l-1-i}(1-d)^i + \varepsilon\sum_{i=l}^{2l-1}d^{2l-1-i}(1-d)^i \tag{4}$$

Eq. (4) can be put in the following form

$$f_{2l-1}(d,\varepsilon) = (1-\varepsilon)f_{1,2l-1}(d) + \varepsilon f_{2,2l-1}(d) \tag{5}$$

where

$$f_{1,2l-1}(d) = \sum_{i=0}^{l-1}\binom{2l-1}{i}d^{2l-1-i}(1-d)^i \tag{6}$$

and

$$f_{2,2l-1} = \sum_{i=l}^{2l-1}\binom{2l-1}{i}d^{2l-1-i}(1-d)^i. \tag{7}$$

The fixed point solutions of Eq. (2) are derived from

$$d = f_{2l-1}(d,\varepsilon) \Leftrightarrow d - f_{2l-1}(d,\varepsilon) = 0 \Leftrightarrow G_{2l-1}(d,\varepsilon) = 0 \tag{8}$$

and

$$G_{2l-1}(d,\varepsilon) = d - f_{2l-1}(d,\varepsilon) \tag{9}$$

Any solution d_0 of Eq. (8)satisfies

$$d_0 = f_{2l-1}(d_0,\varepsilon) \Leftrightarrow d_0 = (1-\varepsilon)f_{1,2l-1}(d_0) + \varepsilon f_{1,2l-1}(d_0) \tag{10}$$

□

Remark 3.1.2. $f_{1,2l-1}(1-d) = f_{2,2l-1}(d)$ *and* $f_{2,2l-1}(1-d) = f_{1,2l-1}(d)$

Proof.

$$f_{1,2l-1}(1-d) = \sum_{i=0}^{l-1}\binom{2l-1}{i}(1-d)^{2l-1-i}d^i = \binom{2l-1}{0}(1-d)^{2l-1} + \binom{2l-1}{1}(1-d)^{2l-1-1}d + ... + \binom{2l-1}{l-1}(1-d)^l d^{l-1}$$

It is known that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \le k \le n$ hence

$$f_{1,2l-1}(1-d) = \binom{2l-1}{2l-1}(1-d)^{2l-1} + \binom{2l-1}{2l-2}(1-d)^{2l-2}d + ... + \binom{2l-1}{l}(1-d)^l d^{l-1} = \sum_{i=l}^{2l-1}\binom{2l-1}{i}d^{2l-1-i}(1-d)^i = f_{2,2l-1}(d)$$

Remark 3.1.3. $f_{1,2l-1}(d) + f_{2,2l-1}(d) = 1$

Proof.

$$\begin{aligned} f_{1,2l-1}(d) + f_{2,2l-1}(d) &= \sum_{i=0}^{l-1} \binom{2l-1}{i} d^{2l-1-i}(1-d)^i + \sum_{i=l}^{2l-1} \binom{2l-1}{i} d^{2l-1-i}(1-d)^i \\ &= \sum_{i=0}^{2l-1} \binom{2l-1}{i} d^{2l-1-i}(1-d)^i = (d+1-d)^{2l-1} = 1^{2l-1} = 1 \end{aligned} \quad (11)$$

□

Putting in the expression of G, Eq. (8), $d_0 \to 1 - d_0$ we get

$$G_{2l-1}(1-d_0, \varepsilon) = 1-d_0-f_{2l-1}(1-d_0, \varepsilon) = 1-d_0-(1-\varepsilon)f_{1,2l-1}(1-d_0)-\varepsilon f_{2,2l-1}(1-d_0) \quad (12)$$

Substituting the expression of d_0 from Eq. (10) in Eq. (12) we get

$$\begin{aligned} G_{2l-1}(1-d_0, \varepsilon) = 1 - (1-\varepsilon)f_{1,2l-1}(d_0) - \varepsilon f_{2,2l-1}(d_0) - \\ (1-\varepsilon)f_{1,2l-1}(1-d_0) - \varepsilon f_{2,2l-1}(1-d_0) \end{aligned} \quad (13)$$

and by remark 3.1.2

$$\begin{aligned} &G_{2l-1}(d_0, \varepsilon) = 1 - (1-\varepsilon)f_{1,2l-1}(d_0) - \varepsilon f_{2,2l-1}(d_0) - \\ &(1-\varepsilon)f_{2,2l-1}(d_0) - \varepsilon f_{1,2l-1}(d_0) = 1 - (f_{1,2l-1}(d_0) + f_{2,2l-1}(d_0)) = 1 - 1 = 0 \end{aligned}$$

□

Proposition 3.1.4. *For each ε, and for constant odd connectivity there is always the solution $d_0 = \frac{1}{2}$.*

Proof.

$$G_{2l-1}(\frac{1}{2}, \varepsilon) = \frac{1}{2} - f_{2l-1}(\frac{1}{2}, \varepsilon) = \frac{1}{2} - (1-\varepsilon)f_{1,2l-1}(\frac{1}{2}) - \varepsilon f_{2,2l-1}(\frac{1}{2}) \quad (14)$$

from remark 3.1.2, putting $d = \frac{1}{2}$ we get that

$$f_{1,2l-1}(\frac{1}{2}) = f_{2,2l-1}(\frac{1}{2}) \quad (15)$$

therefore

$$\begin{aligned} &G_{2l-1}(\frac{1}{2}, \varepsilon) = \frac{1}{2} - (1-\varepsilon+\varepsilon)f_{1,2l-1}(\frac{1}{2}) \\ &= \frac{1}{2} - f_{1,2l-1}(\frac{1}{2}) = \frac{1}{2} - \sum_{i=0}^{l-1} \binom{2l-1}{i} \frac{1}{2}^{2l-1-i} (\frac{1}{2})^i = \\ &= \frac{1}{2} - \sum_{i=0}^{l-1} \binom{2l-1}{i} (\frac{1}{2})^{2l-1} = \frac{1}{2} - (\frac{1}{2})^{2l-1} \sum_{i=0}^{l-1} \binom{2l-1}{i} = \\ &= \frac{1}{2} - (\frac{1}{2})^{2l-1} 2^{2l-2} = \frac{1}{2} - \frac{1}{2} = 0 \end{aligned} \quad (16)$$

□

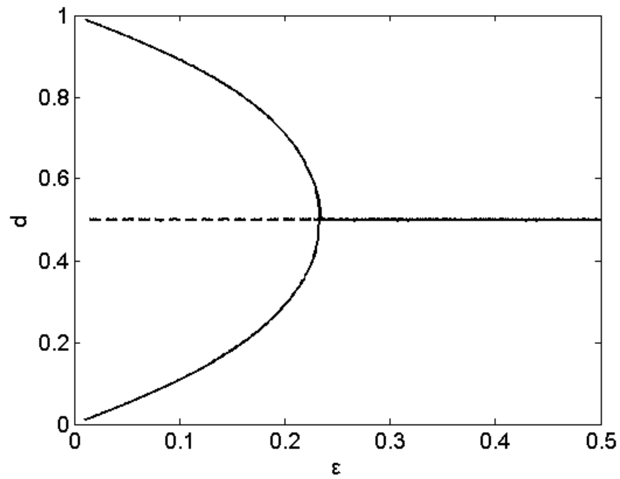

Figure 2: Bifurcation diagram of the density of "A" voters with respect to the switching probability ε, as constructed using the mean field approximation of the majority-voter dynamics evolving one a RRN with a constant degree distribution equal to 5 (as produced using Eq. (8)).

Fig. 2 gives the bifurcation diagram for a random network with constant connectivity degree equal to 5 (self loop is included), resulting from Eq. 8. Clearly there are two branches of symmetric solutions around the steady solution $d_0 = \frac{1}{2}$.

Now suppose that the network has a degree distribution $P(k)$ containing only odd degrees. Specifically, let us assume that the network has N_1 nodes with a degree equal to $2l_1 - 1$, N_2 nodes with a degree equal to $2l_2 - 1$,..., N_k nodes with degree equal to $2l_k - 1, (N_1 + N_2 + ... + N_k = N$ and $l_i \in \ N, i = 1, 2, ...k)$. The time evolution of the density can be split into sums of conditional probabilities of specific degrees:

$$d_{t+1} = f_{tot}(d_t, \varepsilon) \tag{17}$$

and

$$f_{tot}(d_t, \varepsilon) = \sum_{i=1}^{k} f_{2l_i-1}(d, \varepsilon) P(2l_i - 1) \tag{18}$$

The fixed point solution of Eq. (17) is

$$d = f_{tot}(d, \varepsilon) \Leftrightarrow d - f_{tot}(d, \varepsilon) = 0 \Leftrightarrow G_{tot}(d, \varepsilon) = 0 \tag{19}$$

Now we will prove the following

Proposition 3.1.5. *Let (G, E) be a network with a degree distribution containing only odd dergees Then the fixed point equation, Eq. (19), for a constant ε, has symmetric solutions with respect to $d = 1/2$.*

Proof. Suppose that d_0 is a solution of Eq. 19. Then

$$d_0 = \sum_{i=1}^{k} f_{2l_i-1}(d_0, \varepsilon) P(2l_i - 1) \tag{20}$$

From the proof of proposition 3.1.1 we have that for each $i, (i = 1, 2, ..., k)$ the function f_{2l_i-1} can be written as a convex combination of two other functions $f_{1,2l_i-1}$ and $f_{2,2l_i-1}$ (Eq. 5) i.e.

$$f_{2l_i-1}(d, \varepsilon) = (1-\varepsilon) f_{1,2l_i-1}(d) + \varepsilon f_{2,2l_i-1}(d) \tag{21}$$

By remark 3.1.2 we have

$$f_{1,2l_i-1}(1-d) = f_{2,2l_i-1}(d) \tag{22}$$

and

$$f_{2,2l_i-1}(1-d) = f_{1,2l_i-1}(d) \tag{23}$$

Now we set $d_0 \to 1 - d_0$ in the function G_{tot} Eq. (19) to get

$$G_{tot}(1-d_0, \varepsilon) = 1 - d_0 - f_{tot}(1-d_0, \varepsilon) = 1 - d_0 - \sum_{i=1}^{k} f_{2l_i-1}(1-d_0, \varepsilon) P(2l_i - 1) =$$
$$= 1 - d_0 - \sum_{i=1}^{k} ((1-\varepsilon) f_{1,2l_i-1}(1-d_0) + \varepsilon f_{2,2l_i-1}(1-d_0)) P(2l_i - 1) \tag{24}$$

Substituting Eq. (20) in Eq. (24) we get that:

$$\left\{ \begin{array}{c} G_{tot}(1-d_0, \varepsilon) = 1 - \sum_{i=1}^{k} f_{2l_i-1}(d_0, \varepsilon) P(2l_i - 1) \\ - \sum_{i=1}^{k} ((1-\varepsilon) f_{1,2l_i-1}(1-d_0) + \varepsilon f_{2,2l_i-1}(1-d_0)) P(2l_i - 1) \\ = 1 - \sum_{i=1}^{k} ((1-\varepsilon) f_{1,2l_i-1}(d_0) + \varepsilon f_{2,2l_i-1}(d_0)) P(2l_i - 1) \\ - \sum_{i=1}^{k} ((1-\varepsilon) f_{1,2l_i-1}(1-d_0) + \varepsilon f_{2,2l_i-1}(1-d_0)) P(2l_i - 1) \\ = 1 - \sum_{i=1}^{k} ((1-\varepsilon) f_{1,2l_i-1}(d_0) + \varepsilon f_{2,2l_i-1}(d_0) \\ + (1-\varepsilon) f_{2,2l_i-1}(d_0) + \varepsilon f_{1,2l_i-1}(d_0)) P(2l_i - 1) \\ = 1 - \sum_{i=1}^{k} (f_{1,2l_i-1}(1-d_0) + f_{2,2l_i-1}(1-d_0)) P(2l_i - 1) \end{array} \right. \tag{25}$$

Taking into account that $f_{1,2l_i-1}(d) + f_{2,2l_i-1}(d_0) = 1$ and $\sum_{i=1}^{k} P(2l_i - 1) = 1$, the last expression of Eq. (25) reads:

$$G_{tot}(1-d_0, \varepsilon) = 1 - \sum_{i=1}^{k} (f_{1,2l_i-1}(1-d_0) + f_{2,2l_i-1}(1-d_0)) P(2l_i - 1) =$$
$$= 1 - \sum_{i=1}^{k} P(2l_i - 1) = 1 - 1 = 0 \tag{26}$$

□

Fig. 3 gives the bifurcation diagram for a random network with the degree distribution contains only odd degrees, resulting from Eq. (19). The two branches of symmetric solutions around the steady solution $d_0 = \frac{1}{2}$ still exist.

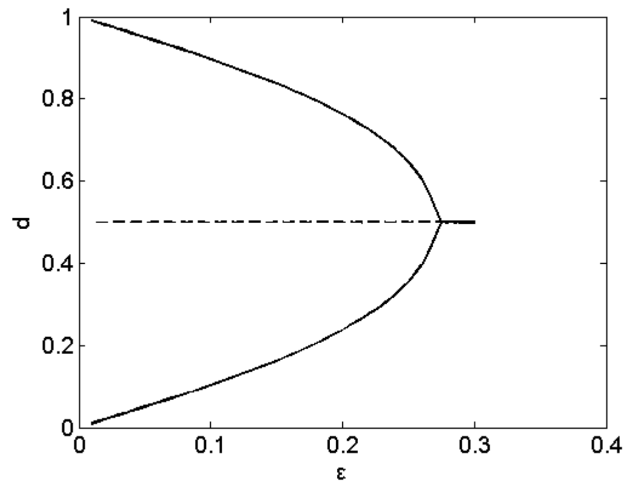

Figure 3: Bifurcation diagram of the density of "A" voters with respect to the switching probability ε, as constructed using the mean-field approximation of the majority-voter dynamics evolving on networks containing only odd degrees (here 5, 7, 9) (as produced using Eq. (18), (19)).

3.2 RRN with even connectivity distribution

In the general case of a network (G, E) with arbitrary even constant connectivity degree, the time evolution of the density reads:

$$d_{t+1} = f_{2l}(d_t, \varepsilon) \tag{27}$$

with

$$f_{2l}(d,\varepsilon) = (1-\varepsilon)\sum_{i=0}^{l-1}\binom{2l}{i}d^{2l-i}(1-d)^i + \varepsilon\sum_{i=l}^{2l}\binom{2l}{i}d^{2l-i}(1-d)^i \tag{28}$$

Eq. (28) can be written as

$$f_{2l}(d,\varepsilon) = (1-\varepsilon)f_{1,2l}(d) + f_{2,2l}(d) + \varepsilon\binom{2l}{l}d^l(1-d)^l \tag{29}$$

where $f_{1,2l}(d) = \sum_{i=0}^{l-1}\binom{2l}{i}d^{2l-i}(1-d)^i$ and $f_{2,2l}(d) = \sum_{i=0}^{l-1}\binom{2l}{i}d^{2l-i}(1-d)^i$. Similar to the proof of remark 3.1.2, the two parts $f_{1,2l}(d)f_{2,2l}(d)$, are symmetric with respect to $d = \frac{1}{2}$. Due to the perturbation term $\varepsilon\binom{2l}{l}d^l(1-d)^l$ the function loses its symmetry, resulting to the symmetry breaking of the corresponding bifurcation diagram of Eq. (27), (Fig. 4).

4 The mean field majority-voter model dynamics in networks with arbitrary degree distribution

In order to extract the mean field approximation for complex networks with arbitrary degree distributions, we choose at random a node i. Let k be the degree of the i-th node

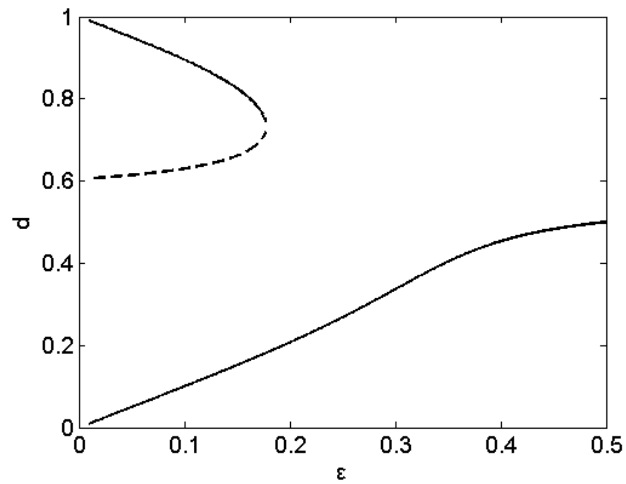

Figure 4: Bifurcation diagram of the density of A voters with respect to the switching probability, as constructed using the mean field approximation of the majority-voter dynamics evolving on a RRN with constant degree distribution equal to 8 (as produced using Eq. (28) and the fixed point solution of Eq. (27)).

and d_t be the density of A voters on the network at time t. Then, the probability of the i-th individual (with degree k) to have a preference A at the next time step $t+1$ is

$$f_k(d_t, \varepsilon) = \sum_{n=0}^{k} a(k, \varepsilon) \binom{k}{n} d_t^{k-n} (1-d_t)^n \tag{30}$$

where $a(k, \varepsilon) = \begin{cases} \varepsilon, if n \leq \frac{k_{max}}{2} \\ 1-\varepsilon, else \end{cases}$. Let $f(d_t, \varepsilon)$ be the probability at time $t+1$ a randomly chosen zero node, to become one. Then

$$f(d_t, \varepsilon) = \sum_{k=1}^{kmax} f_k(d_t, \varepsilon) P(k) \tag{31}$$

where $P(k)$ be the connectivity degree distribution. The time evolution of the switching probability reads

$$d_{t+1} = f(d_t, \varepsilon) \tag{32}$$

For the computation of the stationary points one has to solve the following fixed-point equation:

$$d - f(d, \varepsilon) = 0 \Leftrightarrow G(d, \varepsilon) = 0 \tag{33}$$

For complex networks with a high heterogeneity in the connectivity distribution (such as scale free networks) the fixed point solution of Eq. (33), given Eq. (30) and Eq(31) may become a non-trivial computational task as for a degree distribution with heavy nodes one needs to compute polynomial coefficients of the order of $\binom{k}{n}$.

5 The Equation-free approach for multi-scale computations on complex heterogeneous networks

For detailed individualistic/ stochastic models whose dynamics deploy on heterogeneous networks, the derivation of explicit efficient macroscopic representations for the emergent dynamics in a closed form is most of the times an overwhelming difficult task. The Equation-free approach can be used to bypass the need for extracting explicit continuum models in closed form [29, 14, 44, 36, 30, 43, 25]. The key assumption of the methodology is that a macroscopic model for the emergent dynamics exists and closes in terms of a few coarse-grained variables. These coarse-grained variables are usually the low-order moments of the detailed evolving distribution over the networks. What the methodology does, in fact, is to provide closures on demand in a computational manner. The methodology can be described by the following steps (see also Fig. 5):
(a) Choose the coarse-grained statistics, say $\boldsymbol{x}$, for describing the emergent behavior of the system and an appropriate representation for them (for example the mean value of the underlying evolving distribution).
(b) Choose an appropriate lifting operator μ that maps to a detailed distribution $\mathbf{U}$ on the network. (For example, μ could make random state assignments over the network which are consistent with the densities).
(c) Prescribe a continuum initial condition at a time t_k, say, $\mathbf{x}_{t_k}$.
(d) Transform this initial condition through lifting to N consistent individual-based realizations $\mathbf{U}_{t_k} = \mu \mathbf{x}_{t_k}$.
(e) Evolve these N realizations for a desired time T, generating the $\mathbf{U}_{t_{k+1}}$, where $t_k = kT$.
(f) Obtain the restrictions $x_{t_{k+1}} = \aleph \mathbf{U}_{t_k}$.
The above steps, constitute the so called *coarse timestepper*, which, given an initial coarse-grained state of the system $\mathbf{x}_{t_k}$ at time t_k reports the result of the integration of the model over the network after a given time-horizon T (at time t_{k+1}), i.e.

$$\mathbf{x}_{t_{k+1}} = \Phi_T(\mathbf{x}_{t_k}, \mathbf{p}). \tag{34}$$

where $\Phi_T : R^n \times R^m \to R^n$ having $\mathbf{x}_k$ as initial condition.

The existence of a coarse-grained the temporal evolution operator, Φ_{T_h}, which is assumed to be unavailable analytically in a closed form, implies that the higher order moments of the distributions become, relatively quickly, slaved to the lower, few, slow ones.

At this point one can implement around the coarse-grained input-output map Eq. (34), a fixed point iterative scheme order to compute fixed point or periodic solutions at certain values of the parameter space. For example for low-order systems coarse-grained equilibria can be obtained as fixed points, of the map :

$$\mathbf{x} - \Phi_T(\mathbf{x}, \mathbf{p}) = 0. \tag{35}$$

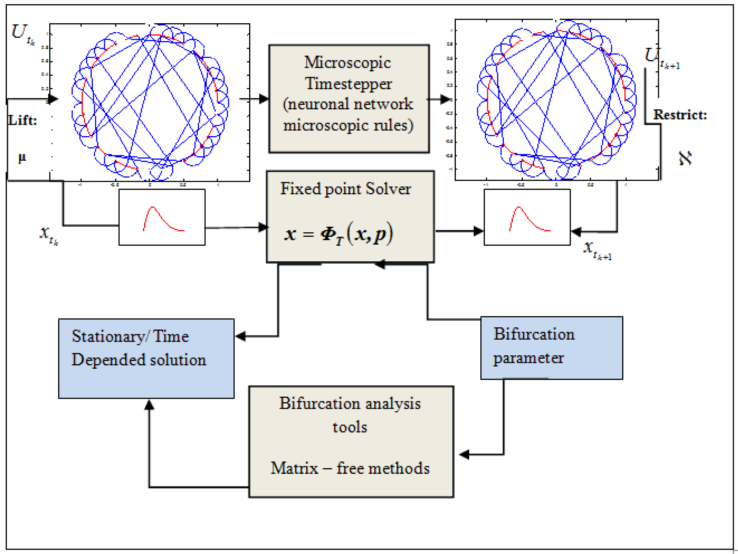

Figure 5: Schematic description of the concept of the Equation-Free approach.

6 Coarse-Grained Numerical analysis using the Equation-free approach

The results are obtained using networks of $N = 10000$ individuals. We performed a coarse-grained analysis for ER, WS and scale-free networks [1, 12, 50, 38]. The coarse-grained bifurcation diagrams, with respect to the switching probability parameter ε, were constructed exploiting the Equation-free framework as described in the previous section. Our coarse-grained variable is the density d of the A voters. At time t_0, we created N_{copies} different distribution realizations consistent with the macroscopic variable d . The coarse timestepper is constructed as the T-map:

$$d_{t+1} = \Phi_T(d_t, \varepsilon) \tag{36}$$

The derived coarse-grained bifurcation diagrams are depicted in Fig. 6-8 respectively. The stationary states on the coarse-grained bifurcation diagram have been obtained as fixed points of Eq. (35) averaging over $N_{copies} = 10000$ realizations. Continuation around the coarse-grained turning points is accomplished by solving the Eq. (35) augmented by the pseudo-arc-length continuation, i.e.:

$$\begin{cases} G(d,\varepsilon) = d - \Phi(d,\varepsilon) = 0 \\ N(D,\varepsilon) = a(d-d_1) + b(\varepsilon-\varepsilon_1) - ds = 0 \end{cases} \tag{37}$$

where $a = \frac{d_1-d_0}{ds}$ and $b = \frac{\varepsilon_1-\varepsilon_0}{ds}$ and is the pseudo arc-length continuation step. The ordered pairs (d_0, ε_0) and (d_1, ε_1) are two already computed solutions. The computation of the fixed points can now be obtained using an iterative procedure like the Newton-Raphson technique.The procedure involves the iterative solution of the following linearized system:

$$\begin{bmatrix} 1 - \frac{\partial \Phi_T}{\partial d} & -\frac{\partial \Phi_T}{\partial \varepsilon} \\ a & b \end{bmatrix} \begin{bmatrix} \delta d \\ \delta \varepsilon \end{bmatrix} = \begin{bmatrix} d - \Phi_T(d,\varepsilon) \\ N(d,\varepsilon) \end{bmatrix} \tag{38}$$

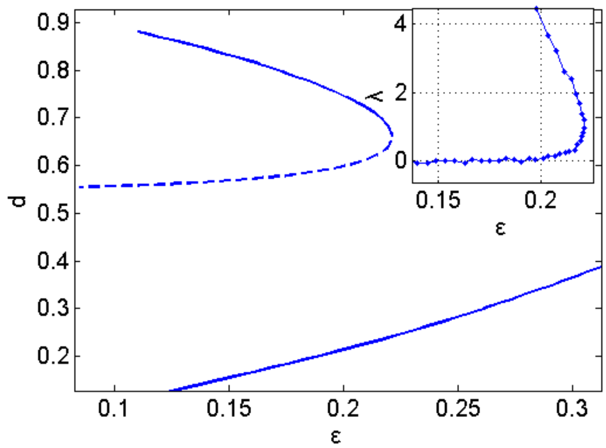

Figure 6: Coarse-grained bifurcation diagram of the density of "A" voters with respect to the switching probability ε, using the detailed majority-voter simulator evolving on an Erdös-Rényi network constructed with connectivity probability $p = 0.0008$, adding, self connection for each node. Solid lines correspond to the coarse-grained stable states while the dotted lines correspond to unstable ones. The inset depicts the computed eigenvalue λ, determining the systems, coarse grained stability.

Note that for the calculation of the Jacobian $\frac{\partial \Phi_T}{\partial d}$ and $\frac{\partial \Phi_T}{\partial \varepsilon}$, no explicit macroscopic evaluation equation are needed. They can be approximated numerically by calling the black-box coarse timestepper at appropriately perturbed values of the corresponding unknowns (d, ε). The above framework enables the microscopic simulator to converge to both coarse-grained stable and unstable solutions and trace their locations [25]. The eigenvalues (here is just one) of the Jacobian $\frac{\partial \Phi_T}{\partial d}$ determine the local stability of the stationary solutions: a fixed point is stable when the modulus of all eigenvalues is smaller than one and unstable if there exists at least one eigenvalue with modulus greater than one.

Each bifurcation diagram consists of two families of solutions. One family of solutions is characterized by a saddle node bifurcation: the high density state (where the majority of individuals vote for A) bifurcates through a turning point (found at $\varepsilon = 0.2207$, $\varepsilon = 0.1869$ and $\varepsilon = 0.1641$, for the networks of ER (Fig. 6), WS (Fig.7) and Barabasi (Fig.8) respectively), marking the change in the stability. The second family of low density states (where the majority of B voters), is stable for all values of ε for all the networks.

In Figs. 9, 10 we also show the bifurcation diagrams obtained using the mean field approximations as these are derived from Eq.(30),(31) and (33) using the ER and WS degree distributions respectively. The degree distributions are taken to be symmetric around $k = 8$.

The coarse-grained bifurcation diagrams obtained using the detailed individual-based model evolving on the ER and WS networks are also shown for comparison reasons. The relatively simple mean field models assume randomly selected individuals with replacement, omitting therefore spatial correlations. Compared to the results obtained by the Equation-free approach, the MF approximation in the case of the ER networks gives

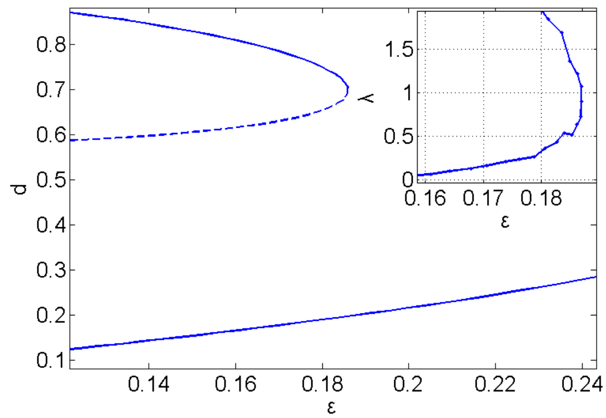

Figure 7: Coarse-grained bifurcation diagram of the density of "A" voters with respect to the switching probability ε, using the detailed majority-voter dynamics simulator evolving on a WS network constructed with rewiring probability $p = 0.2$ and $2k = 8$ initial neighbors, adding self connection for each node. Solid lines correspond to the coarse-grained stable states while the dotted lines correspond to unstable ones. The inset depicts the computed eigenvalue λ, determining the systems, coarse grained stability.

almost identical bifurcation diagrams. In the case of WS the MF gives a qualitatively similar bifurcation diagram, yet a quantitatively different one. In particular, close to the coarse-grained criticalities, the analytical model deviates from the actual detailed simulation results, while the Equation-free framework captures the correct coarse-grained behavior.

7 Conclusions

Over the years, majority-rule or as otherwise called majority-voter models have been extensively used to gain a better understanding on the behavior of many complex systems as diverse as opinion formation and voter/election dynamics, epidemic spread dynamics, culture and language dynamics, crowd flow design and management, and neuroscience. Due to the nonlinear, stochastic nature of such individualistic models and their coupling to complex network structures, the emergent behavior cannot be-most of the times-accurately modeled and analyzed in an efficient straightforward manner. Hence, the systematic exploration of the emergent dynamics of network-evolving individualistic models and in particular those based on the majority-rule mechanism becomes, in this context, of great importance. We proved analytically how the parity and heterogeneity of the degree distributions influences the symmetry of the coarse-grained stationary solutions of the basic majority-voter model.We constructed the mean field approximations of the majority-voter dynamics describing the evolution of the zero-th order moment of the underlying distributions (that is the density of A voters). We also showed how one can exploit the Equation-free framework to bridge the micro to macro scales of the dynamics of stochastic individual-based models that evolve on heterogeneous complex random graphs.

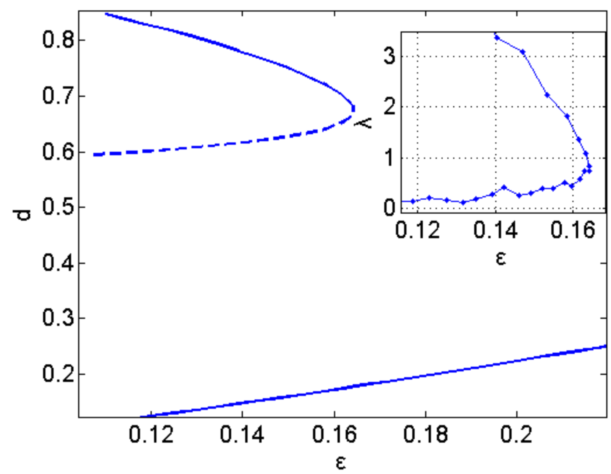

Figure 8: Coarse-grained bifurcation diagram of the density of "A" voters with respect to the switching probability ε, using the detailed majority-voter dynamics simulator evolving on a Barabasi network constructed with $m_0 = 3$ and $m = 2$. Solid lines correspond to the coarse-grained stable states while the dotted lines correspond to unstable ones. The inset depicts the computed eigenvalue λ, determining the systems, coarse-grained stability.

In particular, we showed how systems-level tasks such as bifurcation and stability analysis of the coarse-grained dynamics with respect to network topological characteristics can be performed bypassing the need to extract macroscopic models in a closed form. Our analysis was focused on four of most-cited types of complex networks: Random Regular, ErdsRnyi, Watts and Strogatz and Barabasi (scale free) networks. Using the individualistic, stochastic simulator as a black-box timestepper for the coarse-grained variables, we constructed the coarse-grained bifurcation diagrams with respect to the basic parameter of the majority-voter model: the switching state probability. The derived coarse-grained bifurcation diagrams were compared with the ones obtained using the corresponding mean field approximations. The analysis revealed that especially near the critical turning points the mean-field approximations introduce certain quantitative bias. However the efficiency of the Equation-free approach emerges in the case of scale-free networks. Due to the high heterogeneity of such networks with respect to the heavy tailed connectivity distribution, the bifurcation computations, based on the corresponding mean field approximation, become, as we discussed in section 4, an overwhelming difficult computational task. Further research could be directed towards the investigation of the influence of more accurate closures, such as the ones relating the correlations between the states of two or more connected nodes in the network leading to pairwise approximations [26, 27, 28, 46]. Another issue that could be also further studied is related to one of the basic prerequisites of the Equation-free framework: the a-priori knowledge of the appropriate observables. However, for arbitrary complex networks these are not known before-hand. In this case, the use of state-of-the-art data nonlinear dimensionality reduction techniques that can be exploited to efficiently extract the correct coarse-grained variables from a more complex individual-based large-scale code could be also attempted.

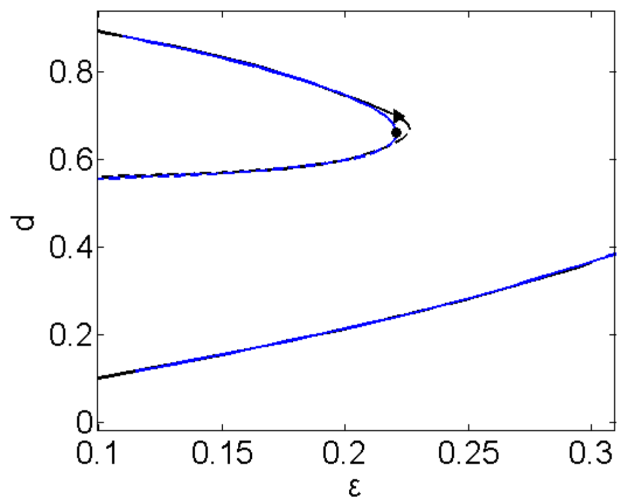

Figure 9: Bifurcation diagram of the density of "A" voters with respect to the switching probability ε, as obtained with the mean-field approximation of the majority-voter dynamics evolving on a Erdös Rényi type network with(marked with a triangle) compared to the coarse-grained bifurcation diagram obtained with the detailed simulator (marked with a circle).

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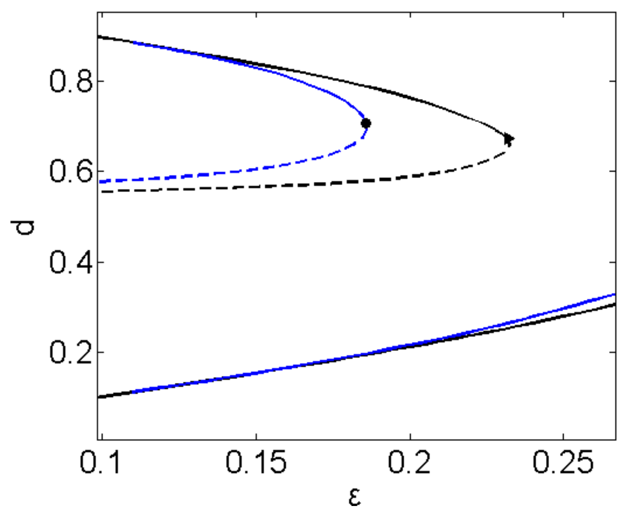

Figure 10: Bifurcation diagram of the density of "A" voters with respect to the switching probability, as obtained with the "mean-field" approximation of the majority-voter dynamics evolving on a WS type network with (marked with a triangle) compared to the coarse-grained bifurcation diagram obtained with the detailed simulator (marked with a circle).

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