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Solving Nonlinear Equations Using Two-Step Optimal Methods

M.A. Hafiz
Department of mathematics, Faculty of Science and arts, Najran University, Saudi Arabia.
e-mail: admin@mhafiz.net

M. S. M. Bahgat
Department of mathematics, Faculty of Science, Minia University, Minia 61111 Egypt.
e-mail: msmbahgat66@hotmail.com

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1 Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x) = 0$. Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method (see [1-9]). In this study we describe new iterative free from second derivative to find a simple root $r$of a nonlinear equation. In the implementation of the method of Noor et al. [10], one has to evaluate the second derivative of the function, which is a serious drawback of these methods. To overcome these drawbacks, we modify the predictor-corrector Halley method by replacing the second derivatives of the function by its suitable finite difference scheme. As we will show, the obtained two-step methods are of fourth-order of convergence and require three evaluations of the function $f(x)$. The procedure of removing the derivatives usually increases the number of functional evaluations per iteration. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $I \approx p^{1/d}$ (see[11]), where $p$is the order of convergence and $d$is the total number of functional evaluations per step. Therefore these methods have efficiency index $4^{1/3} \approx 1.5874$ that is, the new family of methods reaches the optimal order of convergence four, which is higher than $2^{1/2} \approx 1.4142$ of the Steffensen's method (SM) (see [12]), $3^{1/4} \approx 1.3161$of the DHM method (see [13]), $9^{1/5} \approx 1.552$ of the method [14] and our methods are equivalent to the LZM [15] and CTM [16]. We prove that our methods are of fourth-order convergence and present the comparison of these new methods with other methods. Several examples are given to illustrate the efficiency and performance of these methods.

2 Iterative methods

For the sake completeness, we recall Newton, Halley, Traub, and Homeier methods. These methods as follows:

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

It is well known that algorithm 2.1 has a quadratic convergence.

Algorithm 2.2. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

This is known as Halley's method and has cubic convergence [6].

Algorithm 2.3. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= x_n - \tfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \tfrac{f(y_n)}{f'(y_n)}. \end{aligned}$$

Algorithm 2.3 is called the predictor-corrector Newton method (PCN) and has fourth-order convergence (see [16]). Homeier [17] derived the following cubically convergent iteration scheme

Algorithm 2.4. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right).$$

The first and the second derivatives with respect to y, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivatives. This idea plays a significant part in developing our new iterative methods free from first and second derivatives with respect to y. To be more precise, we now approximate $f'(y_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps. Toward this end, an estimation of the function $P_1(t)$ is taken into consideration as follows

$$P_1(t) = a + b(t - x_n) + c(t - x_n)^2$$
$$P_1'(t) = b + 2c(t - x_n)$$

By substituting in the known values

$$P_1(y_n) = f(y_n) = a + b(y_n - x_n) + c(y_n - x_n)^2$$
$$P_1'(y_n) = f'(y_n) = b + 2c(y_n - x_n)$$
$$P_1(x_n) = f(x_n) = a$$
$$P_1'(x_n) = f'(x_n) = b$$

we could easily obtain the unknown parameters. Thus we have

$$f'(y_n) = 2\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) - f'(x_n) = P_1(x_n, y_n) \tag{1}$$

At this time, it is necessary to approximate $f''(y_n)$, with a combination of known values Accordingly, we take account of an interpolating polynomial

$$P_2(t) = a + b(t - x_n) + c(t - x_n)^2 + d(t - x_n)^3$$

and also consider that this approximation polynomial satisfies the interpolation conditions $f(x_n) = P_2(x_n)$, $f(y_n) = P_2(y_n)$, $f'(x_n) = P_2'(x_n)$ and $f'(y_n) = P_2'(y_n)$, By substituting the known values in $P_2(t)$ we have a system of three linear equations with three unknowns. By solving this system and simplifying we have

$$f''(y_n) = \frac{2}{y_n - x_n}\left(\frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x_n)\right) = P_2(x_n, y_n). \tag{2}$$

then algorithm 2.3 can be written in the form of the following algorithm.

Algorithm 2.5. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \tfrac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \tfrac{f(y_n)}{P_1(x_n, y_n)}.$$

This method has fourth-order convergence is called Khattri method (KM) [18]. Now using equations (1) and (2) to suggest the following new iterative methods for solving nonlinear equation, and use Algorithm 2.1 as predictor and Algorithm 2.2 as a corrector. It is established that the following new methods have convergence order four, which will denote by Hafiz and Bahgat Methods (HBM1-HBM5).

HBM1: For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)P_1(x_n, y_n)}{2P_1^2(x_n, y_n) - f(y_n)P_2(x_n, y_n)}.$$

(HBM1) is called the new two-step modified Halley's method free from second and first derivative with respect to y, for solving nonlinear equation $f(x) = 0$.

HBM2: For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left[\frac{1}{f'(x_n)} + \frac{1}{P_1(x_n, y_n)}\right].$$

HBM3: For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{P_1(x_n, y_n)} - \frac{f^2(y)P_2(x_n, y_n)}{2P_1^3(x_n, y_n)}.$$

If $P_2(x_n, y_n) = 0$, then HBM1 and HBM3 deduces Algorithm 2.5.
HBM4: For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right]\frac{f(y_n)}{P_1(x_n, y_n)}.$$

HBM5: For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n + f(y_n)\left[\frac{1}{f'(x_n)} - \frac{4}{f'(x_n) + P_1(x_n, y_n)}\right]$$

Let us remark that, in terms of computational cost, the developed methods (HBM1-HBM5) require only three functional evaluations per step. So, they have efficiency indices $4^{1/3} \approx 1.5874$, that is, the new family of methods (HBM1- HBM5) reaches the optimal order of convergence four, conjectured by Kung and Traub [16].

3 Convergence analysis

Let us now discuss the convergence analysis of the above mentioned methods (HBM1-HBM5).

Theorem 3.1: Let r be a sample zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval I. If $x_0$is sufficiently close to r, then the two-step method defined by (HBM1)has fourth-order convergence.

Proof. Consider to

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

$$x_{n+1} = y_n - \frac{2f(y_n)P_1(x_n, y_n)}{2P_1^2(x_n, y_n) - f(y_n)P_2(x_n, y_n)}. \tag{4}$$

Let r be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r, we get

$$f(x_n) = f(r) + (x_n - r)f'(r) + \frac{(x_n - r)^2}{2!}f^{(2)}(r) + \frac{(x_n - r)^3}{3!}f^{(3)}(r) + \frac{(x_n - r)^4}{4!}f^{(4)}(r) + \cdots,$$

then

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{5}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + \cdots], \tag{6}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(r)}{f'(r)}$, $k = 2, 3, \ldots$ and $e_n = x_n - r$.

Now from (5) and (6), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 + 3c_4)e_n^4 + \cdots, \tag{7}$$

From (3) and (7), we get

$$y_n = r + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + \cdots, \tag{8}$$

From (8), we get,

$$\begin{aligned} f(y_n) &= f'(r)[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + c_4(y_n - r)^4 + \cdots] \\ &= f'(r)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)e_n^4 + \\ &\quad +(4c_5 + 24c_3c_2^2 - 10c_2c_4 - 6c_3^2 - 12c_2^4)e_n^5 + \cdots] \end{aligned} \tag{9}$$

and

$$\frac{f(y_n)}{P_1(x_n, y_n)} = c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_2^3 - 6c_2c_3 + 3c_4)e_n^4 + \cdots \tag{10}$$

$$\frac{P_2(x_n, y_n)}{P_1(x_n, y_n)} = 2c_2 + 4(c_3 - c_2^2)e_n + 2(4c_2^3 - 7c_2c_3 + 3c_4)e_n^2 + \cdots \tag{11}$$

$$\frac{f(y_n)}{P_1(x_n, y_n)}\frac{P_2(x_n, y_n)}{P_1(x_n, y_n)} = 2c_2^2e^2 + 8(c_2c_3 - c_2^3)e_n^3 + 2(11c_2^4 - 21c_2^2c_3 + 6c_2c_4 + 8c_3^2)e_n^4 + \cdots \tag{12}$$

Using equations (8), (9) and (12) in (4), we have :

$$x_{n+1} = r - c_2c_3e_n^4 + O(e_n^5) \tag{13}$$

From (13) and $e_{n+1} = x_{n+1} - r$, we have:

$$e_{n+1} = -c_2c_3e_n^4 + O(e_n^5)$$

which shows that (HBM1) has fourth-order convergence.

Theorem 3.2: Let r be a sample zero of sufficient differentiable function $f : \subseteq R \rightarrow R$ for an open interval I. If $x_0$is sufficiently close to r, then the two-step method defined by (HBM3) has fourth-order convergence.

Proof. Consider to

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{14}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{P_1(x_n, y_n)} - \frac{f^2(y)P_2(x_n, y_n)}{2P_1^3(x_n, y_n)}. \tag{15}$$

Let r be a simple zero of f. Since $f$is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r, we get

$$f(x_n) = f(r)+(x_n-r)f'(r)+\frac{(x_n-r)^2}{2!}f^{(2)}(r)+\frac{(x_n-r)^3}{3!}f^{(3)}(r)+\frac{(x_n-r)^4}{4!}f^{(4)}(r)+\cdots,$$

then

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{16}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + \cdots], \tag{17}$$

where $c_k = \frac{1}{k!}\frac{f^{(2)}(r)}{f'(r)}$, $k = 1, 2, 3, \ldots$ and $e_n = x_n - r$.

Now from (16) and (17), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 + 3c_4)e_n^4 + \cdots, \tag{18}$$

From (14) and (18), we get

$$y_n = r + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + \cdots, \tag{19}$$

From (19), we get,

$$\begin{aligned} f(y_n) &= f'(r)[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + c_4(y_n - r)^4 + \cdots] \\ &= f'(r)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)e_n^4 + \\ &\quad +(4c_5 + 24c_3c_2^2 - 10c_2c_4 - 6c_3^2 - 12c_2^4)e_n^5 + \cdots] \end{aligned} \tag{20}$$

and

$$\frac{f(y_n)}{P_1(x_n, y_n)} = c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_2^3 - 6c_2c_3 + 3c_4)e_n^4 + \cdots \tag{21}$$

$$\left[\frac{f(y_n)}{P_1(x_n, y_n)}\right]^2 = c_2^2e_n^4 + 4(c_2c_3 - c_2^3)e_n^5 + \cdots \tag{22}$$

$$\frac{P_2(x_n, y_n)}{P_1(x_n, y_n)} = 2c_2 + 4(c_3 - c_2^2)e_n + 2(4c_2^3 - 7c_2c_3 + 3c_4)e_n^2 + \cdots \tag{23}$$

$$\frac{1}{2}\left[\frac{f(y_n)}{P_1(x_n,y_n)}\right]^2\frac{P_2(x_n,y_n)}{P_1(x_n,y_n)}=c_2^3e^4+6(c_2^2c_3-c_2^4)e_n^5+\cdots \tag{24}$$

combining (19) - (24), we have :

$$x_{n+1}=r-c_2c_3e_n^4+O(e_n^5)(25)$$

From (25), $e_{n+1}=x_{n+1}-r$and $e_n=x_n-r$, we have:

$$e_{n+1}=-c_2c_3e_n^4+O(e_n^5)$$

which shows that (HBM3) has fourth-order convergence. In Similar way, we observe that the HBM2, HBM4 and HBM5 have also fourth order convergence as follows

$$\begin{gathered}e_{n+1}=c_2^3-3c_4-c_2c_3)e_n^4+O(e_n^5),\ \text{(HBM2)}\\ e_{n+1}=(c_2^3-c_2-c_2c_3)e_n^4+O(e_n^5),\ \text{(HBM4)}\\ e_{n+1}=(3c_2^3-c_2c_3)e_n^4+O(e_n^5).\ \text{(HBM5)}.\end{gathered}$$

4 Numerical examples

For comparisons, we have used the fourth-order Jarratt method [19] (JM) and Ostrowski's method (OM) [11] defined respectively by

$$\begin{aligned}y_n&=x_n-\tfrac{2}{3}\tfrac{f(x_n)}{f'(x_n)}\\ x_{n+1}&=x_n-\left[1-\tfrac{3}{2}\tfrac{f'(y_n)-f'(x_n)}{3f'(y_n)-f'(x_n)}\right]\tfrac{f(x_n)}{f'(x_n)}\end{aligned}$$

and

$$\begin{aligned}y_n&=x_n-\tfrac{f(x_n)}{f'(x_n)}\\ x_{n+1}&=y_n-\tfrac{f(x_n)}{f(x_n)-2f(y_n)}\tfrac{f(y_n)}{f'(x_n)}.\end{aligned}$$

We consider here some numerical examples to demonstrate the performance of the new modified two-step iterative methods, namely (HBM1) - (HBM5). We compare the classical Newton's method (NM), the predictor-corrector Newton method (PCN), Jarratt method (JM), the Ostrowski's method (OM) and the new modified two-step methods (HBM1) - (HBM5), in this paper. In the Tables 1, 2 the number of iteration is $n=5$ for all our examples. But in Table 1 our examples are tested with precision $\varepsilon=10^{-200}$. The following stopping criteria is used for computer programs: $|f(x_{n+1})|<\varepsilon$.

And the computational order of convergence (COC) can be approximated using the formula,

$$COC\approx\frac{\ln|\,(x_{n+1}-x_n)/(x_n-x_{n-1})|}{\ln|\,(x_n-x_{n-1})/(x_{n-1}-x_{n-2})|}$$

Table 1 shows the difference of the root r and the approximation x_n to r, where $r$is the exact root computed with 2000 significant digits, but only 25 digits are displayed for x_n. In Table 1, we listed the number of iterations for various methods. The absolute values of the function $f(x_n)$and the computational order of convergence (COC) are also shown in Tables 2, 3. All the computations are performed using Maple 15. The following examples are used for numerical testing:

$f_1(x) = x^3 + 4x^2 - 10, \quad x_0 = 1\,.$ $\quad f_2(x) = \sin^2 x - x^2 + 1, \quad x_0 = 1.3\,.$
$f_3(x) = x^2 - e^x - 3x + 2, \quad x_0 = 2\,.$ $\quad f_4(x) = \cos x - x, \quad x_0 = 1.7\,.$
$f_5(x) = (x-1)^3 - 1, \quad x_0 = 2.5\,.$ $\quad f_6(x) = x^3 - 10, \quad x_0 = 2\,.$
$f_7(x) = e^{x^2+7x-30} - 1, \quad x_0 = 3.1\,.$

Results are summarized in Table 1, 2 and Table 3 as it shows, new algorithms are comparable with all of the methods and in most cases gives better or equal results.

Table 1. Comparison of Number of iterations for various methods required such that $|f(x_{n+1})| < 10^{-200}$.

Method	f_1	f_2	f_3	f_4	f_5	f_6	f_7
Guess	1	1.3	2	1.7	2.5	2	3.1
NM	9	8	8	8	9	8	10
PCN	5	4	5	5	5	4	5
JM	5	4	5	5	5	4	5
OM	5	4	5	5	5	4	5
HBM1	4	4	5	5	5	4	6
HBM2	5	4	5	5	5	4	5
HBM3	4	4	5	5	5	4	5
HBM4	5	4	6	5	5	4	5
HBM5	5	4	5	5	5	4	5

5 Conclusions

In numerical analysis, many methods produce sequences of real numbers, for instance the iterative methods for solving nonlinear equations. Sometimes, the convergence of these sequences is slow and their utility in solving practical problems quite limited. Convergence acceleration methods try to transform a slowly converging sequence into a fast convergent one. Due to this, paper has aimed to give a rapidly convergent two-point class for approximating simple roots. As high as possible of convergence order was attained by using as small as possible number of evaluations per full cycle. The local order of our class of iterations was established theoretically, and it has been seen that our class supports the optimality conjecture of Kung-Traub [16]. In the sequel, numerical examples have used in order to show the efficiency and accuracy of the novel methods from our suggested second derivative-free class. Finally, it should be noted that, like all other iterative methods, the new methods from the class (HBM1)-(HBM5) have their own domains of validity and in certain circumstances should not be used.

Table 2. Comparison of different methods

Method	x_0	x_5	*COC*	$\lvert x_5 - r\rvert$	$\{\lvert f(x_5)\{\rvert$
f_1	1				
NM		1.36523001341409684576 10286	2	2.1e-9	3.6e-19
PCN		1.36523001341409684576 08068	3.97	1.5e-185	7.1e-746
JM		1.36523001341409684576 08068	3.98	1.5e-185	7.1e-746
OM		1.36523001341409684576 08068	3.99	1.4e-185	7.1e-746
HBM1		1.36523001341409684576 08068	4	2.1e-236	1e-953
HBM2		1.36523001341409684576 08068	3.99	1.4e-185	7.1e-746
HBM3		1.36523001341409684576 08068	4	1.2e-243	1.3e-978
HBM4		1.36523001341409684576 08068	4	3.6e-123	1.2e-495
HBM5		1.36523001341409684576 08068	3.99	1.5e-125	3.1e-505
f_2	1.3				
NM		1.40449164821534122603 50868	2	1.5e-15	4.8e-33
PCN		1.40449164821534122603 50868	4	3.0e-276	8.8e-1109
JM		1.40449164821534122603 50868	4	2.0e-277	1.6e-1113
OM		1.40449164821534122603 50868	4	3.0e-276	8.8e-1109
HBM1		1.40449164821534122603 50868	4	5.0e-339	1.0e-1360
HBM2		1.40449164821534122603 50868	4	3.0e-276	8.8e-1109
HBM3		1.40449164821534122603 50868	4	2.3e-340	5.1e-1366
HBM4		1.40449164821534122603 50868	4	7.0e-275	2.2e-1103
HBM5		1.40449164821534122603 50868	4	6.1e-226	4.8e-907
f_3	2				
NM		0.25753028543986076045 53673	2	9.8e-12	3.4e-25
PCN		0.25753028543986076045 53673	3.99	2.3e-91	5.5e-371
JM		0.25753028543986076045 53673	4	4.1e-93	7.0e-378
OM		0.25753028543986076045 53673	3.99	2.3e-91	5.5e-371
HBM1		0.25753028543986076045 53673	3.99	8.0e-49	8.3e-201
HBM2		0.25753028543986076045 53673	3.99	2.3e-91	5.5e-371
HBM3		0.25753028543986076045 53673	3.99	9.5e-61	1.6e-248
HBM4		0.25753028543986076045 53673	3.95	5.1e-14	2.6e-60
HBM5		0.25753028543986076045 53673	4	2.2e-156	2.9e-231
f_4	1.7				
NM		0.73908513321516064165 53121	1.99	2.3e-14	2.0e-30
PCN		0.73908513321516064165 53121	3.99	2.6e-190	1.9e-766
JM		0.73908513321516064165 53121	3.99	3.5e-196	6.1e-790
OM		0.73908513321516064165 53121	3.99	2.6e-190	1.9e-766
HBM1		0.73908513321516064165 53121	3.99	7.2e-196	6.6e-790
HBM2		0.73908513321516064165 53121	3.99	2.6e-190	1.9e-766
HBM3		0.73908513321516064165 53121	3.99	2.7e-196	6.7e-789
HBM4		0.73908513321516064165 53121	4	5.0e-111	1.2e-448
HBM5		0.73908513321516064165 53121	3.99	9.7e-184	6.9e-740

Table 3. Comparison of different methods

Method	x_0	x_5	*COC*	$\lvert x_5 - r\rvert$	$\lvert f(x_5)\rvert$
f_5	2.5				
NM		2.00000000000000113791023781	2	1.0e-5	3.4e-12
PCN		2	3.99	4.1e-121	5.9e-488
JM		2	3.99	4.1e-121	5.9e-488
OM		2	3.99	4.1e-121	5.9e-488
HBM1		2	4	1.1e-141	1.9e-566
HBM2		2	3.99	4.1e-121	5.9e-488
HBM3		2	4	4.9e-154	6.1e-620
HBM4		2	4	3.3e-175	1.2e-704
HBM5		2	3.99	4.7e-88	3.9e-355
f_6	2				
NM		2.15443469003188372175929356	2	2.2e-16	3.2e-33
PCN		2.15443469003188372175929356	4	1.0e-301	1.3e-1210
JM		2.15443469003188372175929356	4	1.0e-301	1.3e-1210
OM		2.15443469003188372175929356	4	1.0e-301	1.3e-1210
HBM1		2.15443469003188372175929356	4	1.2e-329	1.1e-1322
HBM2		2.15443469003188372175929356	4	1.0e-301	1.3e-1210
HBM3		2.15443469003188372175929356	4	4.2e-330	1.4e-1324
HBM4		2.15443469003188372175929356	4	3.3e-223	7.2e-936
HBM5		2.15443469003188372175929356	4	3.2e-246	4.0e-988
f_7	3.1				
NM		3.00000000000899925734814359	2.03	3.6e-4	1.1e-7
PCN		3	3.99	5.0e-101	7.7e-405
JM		3	3.99	5.2e-98	1.0e-392
OM		3	3.99	5.0e-101	7.7e-405
HBM1		3	3.99	1.1e-46	4.9e-187
HBM2		3	3.99	5.0e-101	7.7e-405
HBM3		3	3.99	1.2e-70	6.0e-283
HBM4		3	3.99	3.0e-102	1.0e-409
HBM5		3	3.99	1.0e-57	8.7e-231

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