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On a Conjecture of Trichotomy and Bifurcation In a Third Order Rational Difference Equation

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Abstract

In this paper it is first investigated for a conjecture of trichotomy of period two for a third order rational difference equation, and then the bifurcation of this equation is further considered. The results obtained partially verify a conjecture in a known literature.

Keywords: Rational difference equation, Trichotomy of period two, Global asymptotic stability, Center manifold; Bifurcation.

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1 Introduction and preliminaries

Consider the following third order rational difference equation

$$x_{n+1} = \frac{x_{n-1} + p}{x_{n-2} + q}, \quad n = 0, 1, \cdots, \tag{1}$$

where the parameters p, q are non-negative real numbers, and the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers.

For Eq. (1), M. R. S. Kulenovic and G. Ladas presented the following question:
Conjecture [3, P. 195] Assume that

$$p, q \in [0,\ \infty).$$

(a) Show that every positive solution of the equation

$$x_{n+1} = \frac{x_{n-1} + p}{x_{n-2} + q}, \quad n = 0, 1, \cdots,$$

converges to a period-two solution if only and if $q = 1$.
(b) Show that when $q > 1$ the positive equilibrium of the equation is global asymptotical stable.
(c) Show that when $q < 1$ the equation possesses positive unbounded solutions.

This question essentially is a conjecture for trichotomy of period two solution. Motivated by this question, our main aim in this paper is to investigate the global behavior of all positive solutions of Eq. (1).

The equilibrium point $\bar{x}$ of Eq. (1) satisfies

$$\bar{x} = \frac{\bar{x} + p}{\bar{x} + q},$$

i.e., $\bar{x}^2 + (q-1)\bar{x} - p = 0$. From this, one can see that Eq. (1) has a unique non-negative equilibrium point

$$\bar{x} = \frac{1 - q + \sqrt{(1-q)^2 + 4p}}{2}. \tag{2}$$

The linearized equation of Eq. (1) associated with this equilibrium is

$$y_{n+1} - \frac{1}{\bar{x} + q} y_{n-1} + \frac{\bar{x}}{\bar{x} + q} y_{n-2} = 0 \tag{3}$$

with the characteristic equation

$$\lambda^3 - \frac{1}{\bar{x} + q}\lambda + \frac{\bar{x}}{\bar{x} + q} = 0. \tag{4}$$

For $q = 1$, the unique non-negative equilibrium point of Eq. (1) reads $\bar{x} = \sqrt{p}$. Eq. (3) and Eq. (4) are respectively reduced into

$$y_{n+1} - \frac{1}{\sqrt{p} + 1} y_{n-1} + \frac{\sqrt{p}}{\sqrt{p} + 1} y_{n-2} = 0 \tag{5}$$

and

$$(\lambda + 1)[(\lambda - \frac{1}{2})^2 + \frac{3\sqrt{p} - 1}{4(\sqrt{p} + 1)}] = 0. \tag{6}$$

Eq. (6) always has one real root $\lambda_1 = -1$, which denotes a period two solution $y_n = (-1)^n$ of the linearized equation (5) and corresponds to a one dimensional local center manifold W^c_{loc} of Eq. (1).

For $p = 0$, three roots of Eq. (6) are -1, 0, 1. The characteristic root $\lambda = 0$ corresponds to a one dimensional (1D) local stable manifold W^s_{loc} of Eq. (1) whereas the unit roots $\lambda = \pm 1$ correspond to a two dimensional (2D) local center manifold W^c_{loc} of Eq. (1).

When $p \in (0, \frac{1}{9}]$, Eq. (6) has another two real roots

$$\lambda_{2,3} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-3\sqrt{p}}{\sqrt{p}+1}}$$

with

$$|\lambda_{2,3}| = |\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-\sqrt{3p}}{\sqrt{p}+1}}| < 1.$$

For $p \in (\frac{1}{9}, \infty)$, Eq. (6) has a pair of conjugate imaginary roots

$$\lambda_{2,3} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3\sqrt{p}-1}{\sqrt{p}+1}}i$$

satisfying

$$|\lambda_{2,3}| = |\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\sqrt{p}}{\sqrt{p}+1}}i| = \frac{1}{2}\sqrt{1+\frac{\sqrt{p}}{\sqrt{p}+1}} < 1.$$

So, for $p \in (0, \infty)$, Eq. (1) always has a 2D local stable manifold W^s_{loc}.

There is always a point of view in engineers and physicians that the local stability of an equilibrium point in a given system implies its global stability.

If this conjecture is true, then the point of view is partly verified. But, we now see that this point of view is not always true.

Because every solution of Eq. (5) converges to either $\bar{x}$ or period two solution, it is natural to conjecture that every solution of Eq. (1) converges to a period two solution for $q = 1$.

When the conjecture is true, the essential changes for the properties of solutions of Eq. (1) will take place at $q = 1$. Namely, the bifurcation of Eq. (1) will occur at $q = 1$. So, the parameter $q = 1$ is a critical point (or bifurcation point).

Generally speaking, given a difference equation

$$x_{n+1} = f(x_n, \mu), \quad n = 0, 1, 2, \cdots,$$

where $x_n \in R^m, \mu \in R^k, f \in C(R^{m+k}, R^m), m, k \in \{1, 2, \cdots\}$ and the initial value $x_0 \in R^m$, its solution is a continuous function with respect to the initial value x_0 and the parameter μ, denoted by $x_n = x(n, x_0, \mu)$. If the change of the initial value x_0 or the parameter μ around a value leads to the essential change of the trajectory structure rule of its solution, then it is implied that a bifurcation of this equation occurs. Correspondingly, the critical value is called to be a bifurcation value. This is similar to the bifurcation of ordinary differential equation.

Certainly, it should be pointed out that the essential change of the trajectory structure rule of a difference equation contains many cases, such as, a solution or an invariant set changes its number, its stability, its boundedness, its period or the cycle length, etc. Therefore, it is meaningful to investigate the bifurcation theory of difference equation according to its own right.

The study of rational difference equation (for short, RDE) is quite challenging and rewarding due to the fact that some results of RDEs offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations; moreover, the investigations of RDEs are still in its infancy so far. To see this, refer to the monographs [1-4] and the papers [5-14] and the references cited therein. Especially, we solved an open problem for the boundedness for the a generalized RDE in [13]; we obtained a result for the global asymptotical stability of a kind of RDE. As a special case, our results solve a conjecture for the global asymptotical stability of a RDE [14].

This rest of this paper is arranged as follows. The global asymptotical stability for the positive equilibrium of Eq. (1) with $q > 1$ is shown in Section 2, which thoroughly solves Conjecture (b).

The investigation for period-two solution of Eq. (1) is formulated in Section 3, partially answering Conjecturer (a) and leaving a gap of $p \in (0, 1)$ to readers. It is shown in Section 4 that this equation possesses positive unbounded solutions when $q < 1$, completely solving Conjecture (c). Synthesizing these results, the center manifold for the equilibrium of Eq. (1) and the analysis of bifurcation are stated in the final Section 5.

2 Global asymptotic stability of positive equilibrium point

For the global asymptotical stability of positive equilibrium point of Eq. (1), one has the following results.

Theorem 2.1 The positive equilibrium point of Eq. (1) is global asymptotical stable for $q > 1$.

Proof For $q > 1$, the linearized equation of Eq. (1) associated with the positive equilibrium (2) is

$$y_{n+1} - \frac{1}{\bar{x}+q} y_{n-1} + \frac{\bar{x}}{\bar{x}+q} y_{n-2} = 0.$$

Obviously, $|-\frac{1}{\bar{x}+q}| + |\frac{\bar{x}}{\bar{x}+q}| = \frac{1+\bar{x}}{\bar{x}+q} < 1$. So, by [2, Remark 1.3.1, P. 13], the positive equilibrium of Eq. (1) is locally asymptotically stable for $q > 1$.

Next, one will show the positive equilibrium of Eq. (1) is globally attractive for $q > 1$. In view of Eq. (1), one can see

$$x_{n+1} < \frac{1}{q} x_{n-1} + \frac{p}{q}, n = 0, 1, \cdots . \tag{7}$$

Denote $n = 2s + t,\ t \in \{0, 1\}$, $y_s = x_{2s+t}$ and $r = \frac{1}{q} \in (0, 1)$.

Then it follows from (7) that

$$y_{s+1} < r y_s + rp, s = 0, 1, \cdots . \tag{8}$$

So, one further gets from (8)

$$y_{s+1} < r^{s+1}y_0 + rp + pr^2 + pr^3 + \cdots + pr^{s+1} = r^{s+1}y_0 + p\frac{r - r^m}{1-r} < y_0 + \frac{pr}{1-r},$$

which indicates that y_s possesses upper bound, say, $R = y_0 + \frac{pr}{1-r}$. And hence so does $\{x_n\}$. Accordingly, from Eq. (1), $x_{n+1} > \frac{p}{R+q}$, namely, $\{x_n\}$ has lower bound.

Therefore,

$$\lim_{n\to\infty} inf x_n = L \quad \text{and} \quad \lim_{n\to\infty} sup x_n = M$$

exist and are finite.

Moreover,

$$0 < L \leq \bar{x} \leq M < \infty.$$

It is clear from Eq. (1) that

$$L \geq \frac{L+p}{M+q}, \quad M \leq \frac{M+p}{L+q}.$$

So, $LM + Lq \geq L + p$ and $ML + Mq \leq M + p$, which implies $(1-q)M + p \geq ML \geq (1-q)L + p$. Therefore, $M \leq L$. Again, $M \geq L$. Hence $M = L$. That is to say, $\lim_{n\to\infty} x_n = \bar{x}$. The proof is complete.

3 Existence of period two solution

In this section one will consider the behavior for prime period two solutions of Eq. (1). The following lemma will be needed [4, P. 12].

Lemma 3.1 Let $F \in C[I^k, I]$ for some interval I of positive real numbers and for some natural number k. Then every positive solution of the equation

$$x_n = F(x_{n-1}, \cdots, x_{n-k}), n = 0, 1, \cdots$$

has a limit in I if the following statements are true:
(1) $F(z_1, \cdots, z_k)$ is nondecreasing in each of its arguments;
(2) $F(z_1, \cdots, z_k)$ is strictly increasing in each of the arguments $z_{i_1}, \cdots, z_{i_e}$, where $i_1, \cdots, i_e$ are relatively prime;
(3) $F(c, c, \cdots, c) = c$ for every $c \in I$.

One has the following result.

Theorem 3.2 (1) If every positive solution of Eq. (1) converges to a period two solution, then $q = 1$.
(2) If $q = 1$, moreover, $p \in \{0\} \cup [1, \infty)$, then every positive solution of Eq. (1) converges to a period two solution.

Proof (1) If every positive solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) converges to a period two solution $\cdots, \alpha, \beta, \alpha, \beta, \alpha, \cdots$, then

$$\alpha = \frac{\alpha + p}{\beta + q}, \quad \beta = \frac{\beta + p}{\alpha + q}.$$

Namely,

$$\begin{cases} \alpha\beta + q\alpha &= \alpha + p, \\ \alpha\beta + q\beta &= \beta + p. \end{cases} \tag{9}$$

Subtracting each other in the above system yields $q(\alpha - \beta) = \alpha - \beta$. Notice that $\alpha \neq \beta$. So, $q = 1$.

(2) Assume $q = 1$. First consider the case $p = 0$. Then Eq. (1) reads

$$x_{n+1} = \frac{x_{n-1}}{x_{n-2} + 1} < x_{n-1}, n = 0, 1, 2, \cdots. \tag{10}$$

Set $n = 2s + t$, $y_s = x_{2s+t}, t \in \{0, 1\}$. Then (10) implies $0 < y_{s+1} < y_s$. Therefore, $\lim_{s\to\infty} y_s$ exists, which implies that both $\{x_{2s}\}$and $\{x_{2s+1}\}$ converge. Denote

$$\lim_{n\to\infty} x_{2n} = \alpha, \lim_{n\to\infty} x_{2n+1} = \beta.$$

Letting n in Eq. (1) be changed into $2n$ and $2n + 1$ respectively and then respectively taking the limits on both sides of Eq. (1) yield

$$\alpha = \frac{\alpha + p}{\beta + q}, \quad \beta = \frac{\beta + p}{\alpha + q}.$$

So, $\cdots, \alpha, \beta, \alpha, \beta, \cdots$ is a period two solution of Eq. (1). Accordingly, $\{x_n\}$ converges to a period two solution (not necessarily prime).

Then consider the case $p \in [1, \infty)$. From Eq. (1), one has $x_n = \frac{x_{n-2}+p}{x_{n-3}+1}$ and so $x_{n-5} = \frac{x_{n-4}+p}{x_{n-2}} - 1$. Thus,

$$x_{n+2} = \frac{x_n + p}{x_{n-1} + 1} = \frac{x_n + p}{\frac{x_{n-3}+p}{x_{n-4}+1} + 1} = \frac{x_n + p}{\frac{\frac{x_{n-5}+p}{x_{n-6}+1}+p}{x_{n-4}+1} + 1} = \frac{x_n + p}{\frac{\frac{\frac{x_{n-4}+p}{x_{n-2}}-1+p}{x_{n-6}+1}+p}{x_{n-4}+1} + 1}. \tag{11}$$

Put $n = 2s + t, t \in \{0, 1\}$ and $y_s = x_{2s+t}, s = -1, 0, \cdots$.. Then, from (11), one can see

$$y_{s+1} = \frac{y_s + p}{\frac{\frac{\frac{y_{s-2}+p}{y_{s-1}}-1+p}{y_{s-3}+1}+p}{y_{s-2}+1} + 1} \triangleq H(y_s, y_{s-1}, y_{s-2}, y_{s-3}), s = 2, 3, \cdots. \tag{12}$$

Evidently, H is increasing with respect to y_s, y_{s-1}, and y_{s-3}. Furthermore, $H(x, x, x, x) = x$ for any $x \in (0, \infty)$.

Next one will show that H is increasing in y_{s-2}. Denote

$$h(x) = \frac{\frac{\frac{x+p}{y}-1+p}{z+1} + p}{x + 1}, x, y, z \in (0, \infty).$$

Then $h'(x) = \frac{(1-p)(1+y)-py(z+1)}{y(z+1)(x+1)^2} < 0$ for $p \geq 1$. That is to say, $h(x)$ is decreasing in x. Thereout, H is increasing with respect to y_{n-2} for $p \geq 1$. Hence, it follows from Lemma 3.1 that every solution $\{y_s\}_{s=-1}^{\infty}$ has a limit and so $\{x_{2s}\}_{s=0}^{\infty}$ and $\{x_{2s+1}\}_{s=-1}^{\infty}$ converge, which indicates that $\{x_n\}$ converges to a period two solution for $q = 1$ and $p \geq 1$. The proof is complete.

Remark 3.3. If it can be proved that every positive solution of Eq. (1) converges to a period two solution for $p = 1$ and $q \in (0, 1)$, then Conjecture (a) will be completely shown. Unfortunately, up to now, this is still an open problem.

4 Existence of unbounded solution

In this section one will investigate the existence of unbounded solutions of Eq. (1) for $q < 1$. The following results are derived.

Theorem 4.1 There exist unbounded solutions of Eq. (1) for $q < 1$.

Proof Consider two cases.

Case 1: $p > 0$. Choose the initial values $x_0, x_{-2} \in (0, 1-q), x_{-1} \geq \frac{p}{x_0} + 1 - q$, which implies $\frac{x_0+p}{x_{-1}+q} \leq x_0$. From $x_{n+1} = \frac{x_{n-1}+p}{x_{n-2}+q}, n = 0, 1, 2, \cdots$, one has

$$x_1 = \frac{x_{-1}+p}{x_{-2}+q} > x_{-1} + p, \qquad x_2 = \frac{x_0+p}{x_{-1}+q} \leq x_0,$$
$$x_3 = \frac{x_1+p}{x_0+q} > x_1 + p > x_{-1} + 2p, \qquad x_4 = \frac{x_2+p}{x_3+q} < \frac{x_0+p}{x_1+q} < x_0,$$
$$x_5 = \frac{x_3+p}{x_2+q} \geq \frac{x_3+p}{x_0+q} > x_3 + p > x_{-1} + 3p, \quad x_6 = \frac{x_4+p}{x_3+q} < \frac{x_0+p}{x_1+q} < x_0.$$

So, inductively, one gets

$$x_{2n+1} > x_{-1} + (n+1)p, \quad x_{2n} < x_0, n = 0, 1, \cdots.$$

Therefore, $lim_{n\to\infty} x_{2n+1} = \infty$, i.e., $\{x_n\}$ is unbounded.

Case 2: $p = 0$. Then by choosing the initial values $x_0, x_{-1}, x_{-2} \in (0, \infty)$ such that $0 < x_0 < x_{-2} < 1 - q, x_{-1} > 1 - q$, one has

$$x_1 = \frac{x_{-1}}{x_{-2}+q} > x_{-1}, \qquad x_2 = \frac{x_0}{x_{-1}+q} < x_0,$$
$$x_3 = \frac{x_1}{x_0+q} > \frac{x_1}{x_{-2}+q} = (\frac{1}{x_{-2}+q})^2 x_{-1}, \qquad x_4 = \frac{x_2}{x_1+q} < \frac{x_0}{x_{-1}+q} < x_0,$$
$$x_5 = \frac{x_3}{x_2+q} > \frac{x_3}{x_{-2}+q} > (\frac{1}{x_{-2}+q})^3 x_{-1}, \qquad x_6 = \frac{x_4}{x_3+q} < \frac{x_0}{x_{-1}+q} < x_0.$$

It follows by induction that $x_{2n+1} > (\frac{1}{x_{-2}+q})^n x_{-1}$ and so $lim_{n\to\infty} x_{2n+1} = \infty$, which implies that $\{x_n\}$ is also unbounded. The proof is over.

5 Analysis of bifurcation

After the above preparations, one will begin to formulate some results for the center manifold of the equilibrium of Eq. (1) and analyze the bifurcation case of Eq. (1).

First, one may transform Eq. (1) to an equivalent system. Let $u_n = x_{n-2}, v_n = x_{n-1}, w_n = x_n$, and $z_n = (u_n, v_n, w_n)^T$. Then Eq. (1) is equivalent to the following system: $z_{n+1} = F(z_n)$, i.e.,

$$\begin{cases} u_{n+1} &= v_n, \\ v_{n+1} &= w_n, \\ w_{n+1} &= \frac{v_n+p}{u_n+q}. \end{cases} \tag{13}$$

The equilibrium point $\bar{z} = (u, v, w)$ of the system (13) satisfies

$$u = v = w = \bar{x} = \frac{1 - q + \sqrt{(1-q)^2 + 4p}}{2}.$$

The Jaccobian matrix of F at the equilibrium point $\bar{z}$ has the form

$$DF(\bar{z}) = \begin{pmatrix} o & 1 & 0 \\ o & 0 & 1 \\ -\frac{v+p}{(u+q)^2} & \frac{1}{u+q} & 0 \end{pmatrix}$$

with the characteristic equation evaluated at the equilibrium point $\bar{z}$

$$\lambda^3 - \frac{1}{\bar{x}+q}\lambda + \frac{\bar{x}}{\bar{x}+q} = 0, \tag{14}$$

which is the same as (4).

The following results may be derived.

Theorem 5.1 Consider the first order 3D system (13). Then the following statements are true.

1. Suppose $q = 1$. If $p = 0$, then the equilibrium point $\bar{z}$ of the system (13) is a center with a 1D local stable manifold and a 2D local center manifold; If $p > 0$, there always is a 2D local stable manifold and a 1D local center manifold at the neighborhood of equilibrium point $\bar{z}$; the latter is a segment of curve L consisting of $\bar{z}$ and the total period two solutions of F, where $L = \{(u, v, w) \in (R^+)^3 | uv = p, u = w\}$, and it is globally asymptotically stable, i.e., the other solutions of (13) regard L as a limit set.
2. For $q > 1$, the equilibrium point $\bar{z}$ of the system (13) is a stable one. There is a 3D stable manifold at the neighborhood of the equilibrium point $\bar{z}$. Namely, the center manifold (2D for $p = 0$ and 1D for $p > 0$) which occurs for $q = 1$ disappears and turns also into a stable manifold (2D for $p = 0$ and 1D for $p > 0$).
3. For $q < 1$, the center manifold becomes an unstable manifold. At this time, except for the orbit $z_n = \bar{z}$ in L, all other orbits on L will tend to infinity along the L.

Proof It is easy to see that the first order 3D system (13) has a unique equilibrium point $\bar{z}$.

1. For $q = 1$, according to the analysis in the introduction in this paper, the characteristic equation (14) of the system (13) has one root $\lambda_1 = -1$. When $p = 0$, the other two roots of (14) is $0, 1$. The characteristic root 0 corresponds a 1D local stable manifold of the equilibrium point $\bar{z}$ of the system (13), and the characteristic roots ± 1 correspond a 2D local center manifold of the equilibrium point $\bar{z}$. When $p \in (0, \frac{1}{9}]$, the equation (14) has two other real roots

$$\lambda_{2,3} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1 - 3\sqrt{p}}{\sqrt{p}+1}}$$

with $|\lambda_{2,3}| < 1$. When $p \in (\frac{1}{9}, \infty)$, the equation (14) has a pair of conjugate imaginary roots

$$\lambda_{2,3} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3\sqrt{p} - 1}{\sqrt{p}+1}}i$$

satisfying

$$|\lambda_{2,3}| = |\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\sqrt{p}}{\sqrt{p}+1}}i| = \frac{1}{2}\sqrt{1+\frac{\sqrt{p}}{\sqrt{p}+1}} < 1.$$

Hence, it is always true that $|\lambda_{2,3}| < 1$ for $p \in (0, \infty)$, which, together with $\lambda_1 = -1$, reads the existence of a 2D local stable manifold and a 1D local center manifold at the neighborhood of equilibrium point $\bar{z}$. The expression of L can be obtained from $z_{n+2} = z_n$.

2. For $q > 1$, the previous Theorem 2.1 tells us the equilibrium point $\bar{z}$ of the system (13) is globally asymptotically stable regardless of $p = 0$ or $p > 0$. This indicates that there is a 3D stable manifold at the neighborhood of the equilibrium point $\bar{z}$. Therefore, the 2D center manifold which occurs for $p = 0$ and $q = 1$ disappears and turns into a 2D stable manifold and the 1D center manifold occurring for $p > 0$ and $q = 1$ disappears and becomes a 1D stable manifold.

3. The correctness follows from Theorem 4.1 stated previously in this paper.

Remark 5.2. It is easily observed that the equilibrium point $\bar{z}$ of the system (13) loses one dimension in the center manifold and gains one dimension in the stable one when $p = 1$ and q crosses the null value. Namely, the dimensional number of center manifold of the equilibrium point $\bar{z}$ of the system (13) varies from 2 to 1. This kind of change for the dimensional number of center manifold of the equilibrium point $\bar{z}$ as the parameter p crosses the null value possibly implies a new mechanism for the creation of bifurcation, which deserves to one's further investigations.

6 Stability of period two solution

The existence of period two solution has been considered for $q = 1$ in above Section 3. When every solution of Eq. (1) converges to a period two solution, how about the stability of the period two solution? We now answer this question.

A period two solution of Eq. (1) or system (13) is a fixed point of $F^2(z) = F(F(z)) = z$ with

$$F^2(z) = F(F(z)) = \begin{pmatrix} w \\ \frac{v+p}{u+q} \\ \frac{w+p}{v+q} \end{pmatrix} \quad \text{for} \quad z = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

The Jaccobian matrix of F^2 has the form

$$DF^2(z) = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{v+p}{(u+q)^2} & \frac{1}{u+q} & 0 \\ 0 & -\frac{w+p}{(v+q)^2} & \frac{1}{v+q} \end{pmatrix}$$

with the characteristic equation evaluated at the period two solution z

$$\lambda^3 - (\frac{1}{u+q} + \frac{1}{v+q})\lambda^2 + \frac{\lambda}{(u+q)(v+q)} - \frac{w+p}{(u+q)^2(v+q)} = 0. \tag{15}$$

By Theorem 3.2 and Theorem 5.1, one can see that there exist period two solutions of Eq. (1) or system (13) only when $q = 1$ and that the period two solution $z^T = (u, v, w) \in (R^+)^3$

satisfies $uv = p$ and $w = u$. Hence, Eq. (15) can be reduced to

$$\lambda^3 - (\frac{1}{u+1} + \frac{u}{u+p})\lambda^2 + \frac{u}{(u+1)(u+p)}\lambda - \frac{up}{(u+1)(u+p)} = 0, \tag{16}$$

where $u > 0$ is a parameter.

Theorem 6.1 Any one period two solution of Eq. (1) with $q = 1$ is unstable.

To prove this conclusion, the following lemma is needed [3, P. 46].

Lemma 6.2 For the equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$ with real coefficients a, b, c, all roots lie inside the unit disk $|\lambda| < 1$ if and only if $|a + c| < 1 + b$, $|a - 3c| < 3 - b$ and $b + c^2 < 1 + ac$.

Proof of Theorem 6.1 Corresponding to (16), the condition

$$\begin{aligned} |a+c| < 1+b \quad &\Leftrightarrow \quad \frac{1}{u+1} + \frac{u}{u+p} + \frac{up}{(u+1)(u+p)} < 1 + \frac{u}{(u+1)(u+p)} \\ &\Leftrightarrow \quad u + p + u(u+1) + up < (u+1)(u+p) + u \\ &\Leftrightarrow \quad 0 < 0. \end{aligned}$$

This is impossible. Hence, there exist at least one root of Eq. (16) not to lie inside the unit disk $|t| < 1$. Thus, Any one period two solution of Eq. (1) with $q = 1$ is unstable.

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References

[1] R. P. Agarwal, "Difference Equations and Inequalities," New York: Marcel Dekker, 1992 (1st edition), 2000 (2nd edition).

[2] V. L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher Order with Applications," Dordrecht: Kluwer Academic Publishers, 1993.

[3] M. R. S, Kulenovic, and G. Ladas, "Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures," Chapman and Hall/CRC, 2002.

[4] E. A, Grove, and G, Ladas, "Periodicites in Nonlinear Difference Equations," Chapman and Hall/CRC, 2005.

[5] R. , DeVault, W. Kosmala, G. Ladas, and S. W. Schaultz, "Global behavior of $y_{n+1} = \frac{p+y_{n-k}}{qy_n+y_{n-k}}$," Nonl. Anal., TMA, 47, 83-89, 2004.

[6] X. Yan, W. Li, and Z. Zhao, "Global asymptotic stability for a higher order nonlinear rational difference equation," Appl. Math. Comput., 118, 1819-1831, 2006.

[7] M. Dehghan, and N. Rastegar, "On the global behavior of a higher-order rational difference equation," Computer Physics Communi., 180(6), 873-878, 2009.

[8] D. Chen, and X. Li, "The bifurcation of cycle length and global asymptotic stability in a rational difference equation with higher order," The Open Appl. Math. J., 2, 80-85, 2008.

[9] X. Li, and D. Zhu, "Two rational recursive sequences," Computers Math. Appl., 47(10/11), 1487-1494, 2004.

[10] X, Li, "Global behavior for a fourth-order nonlinear recursive sequence," J. Math. Anal. Appl., 312(2), 555-563, 2005.

[11] X. Li, "The rule of trajectory structure and global asymptotic stability for a nonlinear difference equation," Appl. Math. Letters, 19, 1152-1158, 2006.

[12] X. Li, and R. P. Agarwal, "The rule of trajectory structure and global asymptotic stability for a fourth-order rational difference equation," J. Korean Math. Soc., 44(4), 787-797, 2007.

[13] X. Li, H. Tang, Y. Liu and G. Xiao, "A conjecture by G. Ladas," Appl. Math.–J. Chinese Univ., (Ser. B), 13 (1), 39-44, 1998.

[14] X. Li, and D. Zhu, "Global asymptotic stability for a nonlinear delay difference equation," Appl. Math.–J. Chinese Univ. (Ser. B), 17(2), 183-188, 2002.