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A Common Fixed Point Theorem of Presic Type for Three Maps in Fuzzy Metric Space

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Abstract

The present paper deals with a common fixed point theorem in Fuzzy metric space by implementing the concept of Presic fixed point theorem [16]. In this paper we have proved a unique common fixed point theorem of Presic type for three maps in a Fuzzy metric space. Also we have obtained the main theorem of R. George [Some fixed point results in dislocated fuzzy metric spaces, Journal of Advanced Studies in Topology, (2012), 3(4), 41-52] as a corollary by employing the conditions of our theorem for dislocated spaces.

Keywords: Fixed point theorem of Presic type, Three maps in Fuzzy metric space

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1 Introduction and Preliminaries

The pioneer concept of fuzzy set theory introduced by Lofti Zadeh [1] of Univ. of California, Berkeley in 1965. There are many view points of the notion of the metric space in fuzzy topology. We can divide them into two groups: First group involved to those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to \mathbb{R}^+$, where X represents the totality of all fuzzy points of a set and satisfies some axioms which are analogous to the ordinary metric axioms. For such approach numerical distances are set up between the fuzzy objects. Where as the second group studied such results in which the distances between objects are fuzzy and the objects themselves may or may not be fuzzy. Erceg [2], Kaleva and Seikkala [4] and Kramosil and Michalek [3] discussed fuzzy metric spaces in detail. Grabiec's [6] proved a fixed point theorem in fuzzy metric space by generalizing the contraction mapping principle due to Banach. Subramanyam [5] generalized Grabiec's result for a pair of commuting maps in the lines of Jungck [8]. Recent definition of Fuzzy Metric spaces credit goes to George and Veermani [7] who modified the concept of fuzzy metric spaces and defined a Hausdorff topology on this space and it has numerous applications in quantum particle physics particularly in connection with both string and E - infinity theory. They have also shown that every metric induces a fuzzy metric in Hausdroff topology.

In this paper we have proved common fixed point theorems of Presic type in Fuzzy metric spaces which extends the results of R. George [14].

We recall the following definitions:

Definition (1) : A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition (2): A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for every $a \in [0, 1]$;

(d) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition (3): A triplet $(X, M, *)$ is said to be a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$;

(1) $M(x, y, t) > 0$,

(2) $M(x, y, t) = 1$ if and only if $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In view of (1) and (2), it is worth pointing out that $0 < M(x, y, t) < 1$ for all $t > 0$, provided $x \neq y$. In view of Definition (1), George and Veermani [7] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric space induces a fuzzy metric space.

In fact, we can fuzzify metric spaces into fuzzy metric spaces in a natural way as is shown

by the following example. In other words, every metric induces a fuzzy metric.
Example (1):Let (X, d) be a metric space and define $a * b = ab$ for all $a, b \in [0, 1]$. Also define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space called standard fuzzy metric space induced by (X, d).

Definition (4): Let $(X, M, *)$ be a fuzzy metric space, then

i) a sequence $\{x_n\}$ in X is said to be convergent to x if

$$lim_{n\to\infty} M(x_n, x, t) = 1,$$

for all $t > 0$;

ii) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$M(x_n, x_m, t) > 1 - \epsilon,$$

for all $t > 0$ and $n, m \geq n_0$;

iii) a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Here, we recall the definition of weakly compatible pair of maps for 2k tuples given by Rao, Kishore and Ali [15].

Definition (5): Let X be a non empty set and $T : X^{2k} \to X, f : X \to X.(f, T)$ is said to be $2k$ - weakly compatible pair, if $f(T(p, p, \cdots, p)) = T(fp, fp, \cdots, fp)$ whenever $p \in X$ such that $fp = T(p, p, \cdots, p)$.

2 Main Rresults

Before proving the main theorem we define the following: Let a function $\phi : [0, 1]^{2k} \to [0, 1]$ such that:

(i) ϕ is an increasing function, i.e., $x_1 \leq y_1, x_2 \leq y_2, \cdots, x_{2k} \leq y_{2k}$ implies $\phi(x_1, x_2, \cdots, x_{2k}) \leq \phi(y_1, y_2, \cdots, y_{2k})$.

(ii) $\phi(t, t, t, \cdots) \geq t$, for all $t \in X$.

(iii) ϕ is continuous in each coordinate variables.

Now we are ready to prove our main:
Theorem Let $(X, M, *)$ be a Fuzzy Metric space and $S, T : X^{2k} \to X, f : X \to X$ be mappings satisfying for each positive integer k :

$$M(S(x_1, x_2, \cdots, x_{2k-1}, x_{2k}), T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), qt) \geq \phi\{M(fx_i, fx_{i+1}, t) : 1 \leq i \leq 2k\} \quad (1)$$

for all $x_1, x_2, \cdots, x_{2k+1} \in$ X, $0 < q < \frac{1}{2}$ and $t \in [0, \infty)$;

$$M(T(y_1, y_2, \cdots, y_{2k-1}, y_{2k}), S(y_2, y_3, \cdots, y_{2k}, y_{2k+1}), qt) \geq \phi\{M(fy_i, fy_{i+1}, t) : 1 \leq i \leq 2k\} \quad (2)$$

for all $y_1, y_2, \cdots, y_{2k+1} \in X$; and

$$M(S(u,u,\cdots,u), T(v,v,\cdots,v), qt) > M(fu, fv, t). \tag{3}$$

for all $u, v \in X$, with $u \neq v$. Suppose that $f(X)$ is complete and either (f,S) or (f,T) is $2k$-weakly compatible pair. Then there exists a unique point p in X such that $fp = p = S(p,p,\cdots,p) = T(p,p,\cdots,p)$.

Proof: Suppose $x_1, x_2, \cdots, x_{2k}$ are arbitrary points in X and for $n \in \mathbb{N}$.
Define $fx_{2k+2n-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, \cdots, x_{2n+2k-2})$
and
$fx_{2k+2n} = T(x_{2n}, x_{2n+1}, x_{2n+2}, \cdots, x_{2n+2k-1})$.
Let $\alpha_n = M(fx_n, fx_{n+1}, qt)$.

Claim: $\alpha_n \geq (\frac{K-\theta^n}{K+\theta^n})^2$ for all $n \in \mathbb{N}$, where $\theta = \frac{1}{2q}$ and $K = min\{\frac{\theta(1+\sqrt{\alpha_1})}{1-\sqrt{\alpha_1}}, \frac{\theta^2(1+\sqrt{\alpha_2})}{1-\sqrt{\alpha_2}}, \cdots, \frac{\theta^{2k}(1+\sqrt{\alpha_{2k}})}{1-\sqrt{\alpha_{2k}}}\}$.

By selection of k we have $\alpha_n \geq (\frac{K-\theta^n}{K+\theta^n})^2$ for $n = 1, 2, \cdots, 2k$.

Now

$$\begin{aligned}\alpha_{2k+1} &= M(fx_{2k+1}, fx_{2k+2}, qt)\\ &= M(S(x_1, x_2, \cdots, x_{2k-1}, x_{2k}), T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), qt)\\ &\geq \phi\{M(fx_i, fx_{i+1}, t) : i = 1, 2, \cdots, 2k\};\end{aligned}$$

by (1), we have

$$\begin{aligned}\alpha_{2k+1} &\geq \phi\{\alpha_1, \alpha_2, \cdots, \alpha_{2k-1}, \alpha_{2k}\}\\ &\geq \phi\{(\frac{K-\theta^1}{K+\theta^1})^2, (\frac{K-\theta^2}{K+\theta^2})^2, \cdots, (\frac{K-\theta^{2k-1}}{K+\theta^{2k-1}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2\}\\ &\geq \phi\{(\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, \cdots, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2\}\\ &\geq (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2\\ &\geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\end{aligned}$$

Thus $\alpha_{2k+1} \geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2$.

Similarly,we have

$$\begin{aligned}\alpha_{2k+2} &= M(fx_{2k+2}, fx_{2k+3}, qt)\\ &= M(T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), S(x_3, x_4, \cdots, x_{2k+2}), qt)\\ &\geq \phi\{M(fx_i, fx_{i+1}, t) : i = 2, 3\cdots, 2k+1\} \quad \text{by} \quad (1)\\ &= \phi\{\alpha_i : i = 2, 3, \cdots 2k+1\}\\ &\geq \phi\{(\frac{K-\theta^2}{K+\theta^2})^2, (\frac{K-\theta^3}{K+\theta^3})^2, \cdots, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\}\\ &\geq \phi\{(\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, \cdots, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\}\\ &\geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\\ &\geq (\frac{K-\theta^{2k+2}}{K+\theta^{2k+2}})^2\end{aligned}$$

Thus $\alpha_{2k+2} \geq (\frac{K-\theta^{2k+2}}{K+\theta^{2k+2}})^2$. Then our claim is true.

Now we show that the sequence x_n is a Cauchy sequence in X. For any $n, p \in \mathbb{N}$, we have

$$\begin{aligned} M(fx_n, fx_{n+p}, t) &\geq M(fx_n, fx_{n+1}, \frac{t}{2}) * M(fx_{n+1}, fx_{n+2}, \frac{t}{2^2}) * \cdots * M(fx_{n+p-1}, fx_{n+p}, \frac{t}{2^p}) \\ &\geq \alpha_n * \alpha_{n+1} * \cdots * \alpha_{n+p-1} \\ &\geq (\frac{K-2^n}{K+2^n})^2 * (\frac{K-2^{2n}}{K+2^{2n}})^2 * \cdots * (\frac{K-2^{np}}{K+2^{np}})^2 \\ &\to 1 * 1 * \cdots * \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{fx_n\}$ is Cauchy sequence. Since $f(X)$ is a complete sub space of X, then there exists z in $f(X)$ such that $lim_{n\to\infty} fx_n = z$. For z in $f(X)$, there exists $p \in X$ such that $z = fp$. Then for any integer n, using (1) and (2), we have

$$M(S(p,p,p\cdots,p), fp, t) = lim_{n\to\infty} M(S(p,p,\cdots,p), fx_{2n+2k-1}, t)$$

$$= lim_{n\to\infty} M(S(p,p,\cdots,p), S(x_{2n-1}, x_{2n}, \cdots, x_{2n+2k-2}), t)$$

$$\geq lim_{n\to\infty} M(S(p,p,\cdots,p), T(p,p,\cdots,x_{2n-1}), \tfrac{t}{2})$$

$$*M(T(p,p,\cdots,x_{2n-1}), S(p,p,\cdots,x_{2n-1},x_{2n}), \tfrac{t}{2^3}) * \cdots$$

$$*M(T(p,x_{2n-1},\cdots,x_{2n+2k-3}), S(x_{2n-1},x_{2n},\cdots,x_{2n+2k-2}), \tfrac{t}{2^{k-1}})$$

$$\geq lim_{n\to\infty} \phi\{M(fp,fp,t), M(fp,fp,t), \cdots, M(fp, fx_{2n-1}, t)\}$$

$$*\phi\{M(fp,fp,t), M(fp,fp,t), \cdots, M(fx_{2n-1}, fx_{2n}, t)\} * \cdots$$

$$*\phi\{M(fp,fx_{2n-1},t), M(fx_{2n-1},fx_{2n},t), \cdots, M(fx_{2n+2k-3}, fx_{2n+2k-2}, t)\}$$

$\to 1$,
i.e., $M(S(p,p,\cdots,p), fp, t) = 1$ and so $c(f,T) \neq \Phi$ where $c(f,T)$ denote the set of all coincidence points of the mappings f and T.
So that

$$S(p,p,\cdots p) = fp \tag{4}$$

Consider

$$\begin{aligned} M(fp, T(p,p,\cdots,p), t) &= M(S(p,p,\cdots,p), T(p,p,\cdots,p), t) \\ &\geq \phi\{M(fp,fp,t), M(fp,fp,t), \cdots, M(fp,fp,t)\} \\ &\geq M(fp,fp,t) \\ &= 1 \end{aligned}$$

Thus

$$T(p,p,\cdots,p) = fp. \tag{5}$$

Now suppose that (f, S) is $2k$ - weakly compatible pair. Then we have
$f(S(p, p, \cdots, p)) = S(fp, fp, \cdots, fp)$.
$f^2p = f(fp) = f(S(p, p, \cdots, p)) = S(fp, fp, \cdots, fp)$.
Suppose that $fp \neq p$. Then from (3), we have

$$\begin{aligned} M(f^2p, fp, t) &= M(S(fp, fp, \cdots, fp), T(p, p, \cdots, p), t) \\ &\geq \{M(f^2p, fp, t), M(f^2p, fp, t), \cdots, M(f^2p, fp, t)\} \\ &\geq M(f^2p, fp, t). \end{aligned}$$

It is a contradiction.
Therefore $fp = p$. Now from (4) and (5), we have
$fp = p = S(p, p, \cdots, p) = T(p, p, \cdots, p)$.
Uniqueness of p: Suppose there exists a point $q \neq p$ in X such that
$fq = q = S(q, q, \cdots, q) = T(q, q, \cdots, q)$
Consider

$$\begin{aligned} M(fp, fq, t) &= M(S(p, p, \cdots, p), T(q, q, \cdots, q), t) \\ &\geq \phi\{M(fp, fq, t), M(fp, fq, t), \cdots, M(fp, fq, t)\} \quad \geq M(fp, fq, t). \end{aligned}$$

It is a contradiction. Therefore $p = q$.
When $S = T$ and $2k$ is replaced by k in main theorem, we get the following corollary.
Corollary(1): Let $(X, M, *)$ be a FM - space, k is a positive integer $T : X^k \to X, f : X \to X$ be mappings satisfying

$$M(T(x_1, x_2, \cdots, x_k), T(x_2, x_3, \cdots, x_{k+1}, qt) \geq \phi\{M(fx_i, fx_{i+1}, t) : 1 \leq i \leq 2k\}$$

for all $x_1, x_2, \cdots, x_{k+1}$ in $X, 0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

$$M(T(u, u, \cdots, u), T(v, v, \cdots, v), qt) > M(fu, fv, t), \forall u, v \in X \quad \text{with} \quad u \neq v$$

Suppose that $f(X)$ is complete and (f, T) is k - weakly compatible pair. Then there exists a unique point $p \in X$ such that $fp = p = S(p, p, \cdots, p) = T(p, p, \cdots, p)$.

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