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Chaotic Dynamical Behavior of Recurrent Neural Network

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Abstract

On account of their role played in the fundamental biological rhythms and by considering their potential use in information processing, the dynamical properties of an artificial neural network are particularly interesting to investigate. In order to reduce the degree of complexity of this work, we have considered in this paper a fully connected neural network of two discrete neurons. We have proceeded to a qualitative and quantitative study of their state evolution by means of numerical simulation. The first aim was to find the possible equilibrium states. Other authors have already shown that some oscillating state can occur. So, the second aim was to analyze the dynamical properties of each of them. We have computed the value of the Lyapunov's exponents and the fractal dimension. The sensitivity of the dynamical characteristics to parameters such as the weights of the connections and the shape of the activation function has been studied.

Keywords: Bifurcation, Chaotic behaviour, Dynamical Systems, Neural Oscillator, Recurrent Neural Networks, Strange attractors.

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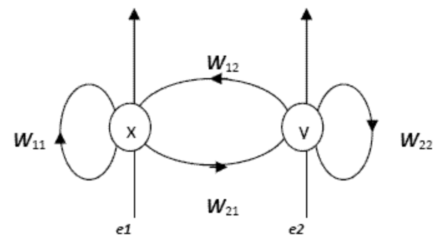

Figure 1: Neural network.

1 Introduction

Chaotic dynamical behavior in the brain has recently been observed and discussed. Whether chaotic behavior of neural networks has any application in biological modeling such us learning and information processing is a very controversial issue [1]. An important question is the biological implications of chaos in neural networks [11, 16]. Neverthles, a lot of effort has gone into modeling and analyzing the chaotic behavior of biological systems, especially for the Hodgkin -Huxley axon model [22]. In this paper, we have tried to study this phenomenon by using a recurrent neural network, which we find to be conceptually simpler, and more appropriate for dealing with discrete or symbolic data [21]. This recurrent neural network is constituted of only two neurons with the sigmoïdal neuron activation function and has no external inputs and no time delay. Our practical objective to studied this kind of network is to design network dynamics for performing a given task by adjusting network parameters. This involves two major issues in dynamical system theory, namely, dynamics and bifurcation. Dynamics is concerned with the asymptotic behavior of the networks, which includes limit sets (e.g., fixed points, periodic orbits and chaotic invariant sets) and their asymptotic stability (e.g., stable, unstable and saddle), while bifurcation is concerned with how the dynamics changes as parameters are varied [24]. In this context, we have performed numerical simulations for determinate the equilibrium states and their asymptotic stability in the first time, and measure the chaos by evaluating the Lyaounov's exponents, as well as the fractal dimension in the second time. The influence of the parameters of the system on the dynamic behavior has been also studied.

2 Model and equations

The study proposed network is a variant of the Mac-Culloch and Pitt's model, where the active parts are two discrete neurons. The first is an excitatory neuron, and the second is an inhibitory one. The network is fully connected and we assign a weight w_{ij} to each connection (Fig 1). The representative state evolution of this model is given by the system of equations (1).

$$\begin{cases} x(t+1) = f(w_{11}x(t) + w_{12}y(t) + e_1) \\ y(t+1) = f(w_{21}x(t) + w_{22}y(t) + e_2) \end{cases} \tag{1}$$

Where x and y are the state variables of the two neurons, their values lie in the range $I = [0\ 1]$. The synaptic weights are w_{ij}. Some external inputs e_1 and e_2 have their value assigned to zero in a first step.

The activation function f has a sigmoïdal form, and is called Fermi's function. It is given by the following equation:

$$f(z) = \frac{1}{1 + e^{-4\sigma z}} \tag{2}$$

Where σ is the constant of Fermi or neuron gain . In this paper, we have specially considered this constant on account of its contribution to the activation function and the dynamical sensitivity of the system [22]. In fact:

- It controls the maximal slope of the function f : the larger the value of σ is, the closer the function f approximates the step function.
- It serves a purpose (at least mathematically) that a change of σ causes a change of all connectivity weights w_{ij} and therefore affects the dynamic behavior of the network. The effect of an increasing or decreasing value of σ is similar to the effect of learning modifying the weights.

For this network it has been proven mathematically that with an increasing fermi constant σ the period doubling route is leading to chaos [22]. In particular, it shows chaotic behavior for the parameter setting used here, $\sigma = 1$ [12, 22]. See also the diagrams of bifurcation (figure 5).

3 Equilibrium states

To determinate the equilibrium states, we assume that a such state exists, so:

$$x_i(k+1) = x_i(k) \tag{3}$$

This condition applied to the components of the state variables of the studied system (1) gives:

$$\begin{cases} x(k+1) = x(k) \\ y(k+1) = y(k) \end{cases} \tag{4}$$

It follows that the two algebraic equations shows in (5) must be simultaneously solved:

$$\begin{cases} f(w_{11}x(k) + w_{12}y(k)) = x(k) \\ f(w_{21}x(k) + w_{22}y(k)) = y(k) \end{cases} \tag{5}$$

As the function f is a non-linear function, the analytical solving of this system of equations is not considered for the time being [7, 6]. A graphical method of solving is applied. This method consists to draw respectively on the same reference the following graphs: $x(k)$ vs $y(k)$ and $y(k)$ vs. $x(k)$. The possible points of intersection correspond to the sought equilibrium states (Fig 2).

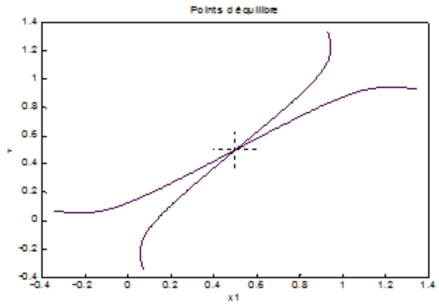

Figure 2: Equilibrium state.

To carry out this task, the determination of the elements of the synaptic matrix W is necessary, because they are as relevant as the neuron gain σ and that affects directly the dynamic behavior of the system. The choice of this matrix is arbitrary and does not follow a general law. As usual, we consider that the mutual connection between the neurons are symmetrical, and $w_{ij} = w_{ji}$. This implies that the matrix is symmetrical. This assumption is funded in the case where the learning law of Hebb is taking account. During the process of learning, the mutual connections lead to reach the same value for the weights. More, the symmetrical form of W leads to some important properties. But if we consider the biological aspect, the same value for the mutual connection is not imperative itself. Some biological measurements have shown that the excitatrice ($w_{ij} > 0$), or inhibitrice ($w_{ij} < 0$) synaptic property depends only of the neuron j, transmitter of signal [10]. Unfortunately, the asymmetrical networks are much more difficult to analyze, what explains the restricted number of works which were dedicated to them [17].

In this work, we are interested in the study of the behavior and the properties of the biologic nervous system from an already established model, therefore the considered network is not a processing system whose objective is to carry out a precise task. So, the phase of learning is not necessary to establish the weight matrix.

In the previous works, one finds several approaches to establish this matrix. Sompolinsky [15,19], for example, demonstrated that continuous-time networks with random asymmetrical connection will be chaotic asymptotically as number of neurons n tends to infinity, provided that the origin is not a stable fixed point. But it is not clear how "spontaneous" chaos occurs an autonomous neural networks of finite neurons with no time delay and no external inputs. On the other hand, X.Wang [22, 4, 8] proved, that for a certain class of connection weight matrices, the simple neural network is dynamically equivalent to a one-parameter full family of S-unimodal maps on the interval [0 1] ,which is well-known to become chaotic through the periodic-doubling route as the parameter varies. In this paper we use this mathematical property, which seems to us to be convenient for this study.

We consider now the following matrix:

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \tag{6}$$

During the stationary state of the network, the values of the weights stabilize, and we can show that:

$$\left\{ \begin{array}{l} w_{11} = -w_{12} \\ w_{21} = -w_{22} \end{array} \right. , \tag{7}$$

letting $w_{11} = a$ and $w_{21} = b$, one obtains the following new matrix:

$$W = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix} \tag{8}$$

According to [22, 18], the dynamic evolution of the network becomes chaotic if one chooses $b < a < 0$ with $\frac{b}{a} > 2$ (see figure 5).

4 Study of the stability

The figure 2 shows a single intersection point in the interval [0 1], corresponding to the equilibrium state. Its coordinates in the space phase are $(x_e, y_e) = (0.5, 0.5)$. To analyse the condition of stability in the vicinity of this equilibrium state, we consider the effect of a perturbation and we study the resulting evolution [5]. Let:

$$\overrightarrow{X}(k+1) = \overrightarrow{X}(k) \text{ at the equilibrium state with } \overrightarrow{X}(k) = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \tag{9}$$

Or:

$$\overrightarrow{X}(k+1) = \overrightarrow{F}(\overrightarrow{X}), \qquad \overrightarrow{F} = \left\{ \begin{array}{l} f_1 \\ f_2 \end{array} \right. \tag{10}$$

Let us consider a small increment $\overrightarrow{\delta X}(k)$, such as:

$$\overrightarrow{\delta X}(k) = [\delta x(k), \delta y(k)] \tag{11}$$

So, we may write for each components of the state:

$$\left\{ \begin{array}{l} \delta x(k+1) = f_1 \left[x(k) + \delta x(k), y(k) + \delta y(k) \right] - f_1 \left[x(k), y(k) \right] \\ \delta y(k+1) = f_2 \left[x(k) + \delta x(k), y(k) + \delta y(k) \right] - f_2 \left[x(k), y(k) \right] \end{array} \right. \tag{12}$$

Thus, approximately:

$$\left\{ \begin{array}{l} \delta x(k+1) = \dfrac{\partial f_1}{\partial x} \delta x(k) + \dfrac{\partial f_1}{\partial y} \delta y(k) \\ \delta y(k+1) = \dfrac{\partial f_2}{\partial x} \delta x(k) + \dfrac{\partial f_2}{\partial y} \delta y(k) \end{array} \right. \tag{13}$$

Or:

$$\overrightarrow{\delta X}(k+1) = J \cdot \overrightarrow{\delta X}(k) \text{ with } J = \begin{bmatrix} \dfrac{\delta f_1}{\delta x} & \dfrac{\delta f_1}{\delta y} \\ \dfrac{\delta f_2}{\delta x} & \dfrac{\delta f_2}{\delta y} \end{bmatrix} \tag{14}$$

Where J is the Jacobian matrix of the linearized transformation.

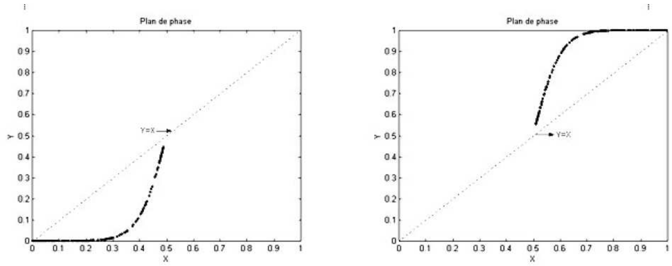

Figure 3: Dynamical behavior.

The nature of stability depends then on the eigenvalue of the jacobian matrix J evaluated at the equilibrium state. In our case the characteristic equation can be write:

$$\lambda^2 - Tr \cdot \lambda + \Delta = 0 \ avec \ \begin{cases} Tr = \dfrac{\delta f_1}{\delta x} + \dfrac{\delta f_2}{\delta y} \\ \Delta = \dfrac{\delta f_1}{\delta x} \cdot \dfrac{\delta f_2}{\delta y} - \dfrac{\delta f_1}{\delta x} \cdot \dfrac{\delta f_2}{\delta y} \end{cases} \tag{15}$$

where λ is a eigenvalue, T_r is the trace of J and Δ its determinant.

The evolution becomes stable if and only if the each eigenvalue module is inferior to the unit.

By using the matrix (7), one can show that the equilibrium point of the system becomes stable for $b - \frac{1}{4} < a < b + \frac{1}{4}$, and unstable outside this interval (a and b are the synaptic matrix elements). To obtain the oscillation in the system it is necessary to choose a and b such manner that previous inequality is not verified, in the opposite case the system converges to the unstable equilibrium point.

5 Dynamical behavior

So as to undertake a qualitative study allowing the measure of chaos in our network, we consider the system (1) with $e_1 = e_2 = 0$. As is shown in the figure 3, by using different values of synaptic weights the system represents oscillations between 0 and 1 [12, 23] (Fig. 3).

Accordingly, in the two-dimensional phase space (y vs x) an attractor can be observed , as is shown in figure 4 (left) for the start vectors $(x_0, y_0) = (0.4, 0.3999)$. However, for a slightly different choice of start vector, $(x_0, y_0) = (0.3999, 0.4)$, the system converges to another attractor, as shown in figure 4 (right). These two attractors divide phase space into two separate regions, their respective basins of attraction. These are separated along the line $y = x$. Start vectors $(x_0, y_0) = (a, b)$ with $a > b$ converge to the lower left attractor, whereas those with $a < b$ converge to the attractor on the upper right. Start vectors $(x_0, y_0) = (a, a)$ lying on the line $y = x$ converge to the unstable fixed point $(0.5, 0.5)$. This sensitivity to initial conditions (start vectors) which we notice in the behavior of the system characterizes generally chaotic dynamical systems [3].

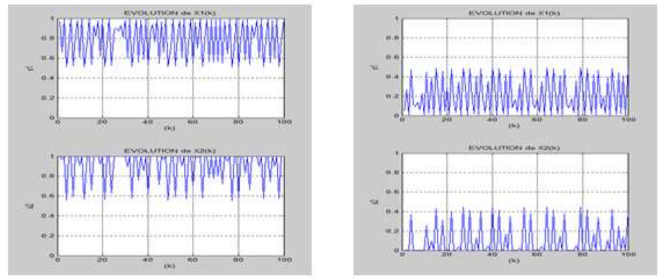

Figure 4: Attractors.

5.1 Bifurcation

To be able to analyze more precisely the behavior of the system, when a parameter varies, we have calculated the bifurcation diagrams, for different values of neuron gain and synaptic weights. The principle and the meaning of these diagrams are explained hereafter [14].

Considering a discrete system of order n, $x_n(k+1) = f(x_n(k), \alpha)$ with a parameter $\alpha \in R$. When α varies, the limit sets of the system change equally. Typically, a weak modification of α product a weak qualitative change of the limit set. For example to disturb α could change slightly the position of the limits set, and if this set is not a equilibrium point, its form or its size could modify equally. A such change is called bifurcation, and the value of α for which this bifurcation appears is called bifurcation value. To study the appearance of bifurcation in a system, one draws then a bifurcation diagram. A bifurcation diagram is a graph representing the position of system's equilibrium points, when the parameter α varies.

The bifurcation diagrams which are present here, have been calculated by report to the neuron gain and to synaptic weights. For each σ (resp w_{ij}) we have taken 600 iterations for each point with initial vector $(x_0, y_0) = (0,3999, 0,4)$ and we have obtained some graphs represented in the figure 5.

5.2 Spectral analysis

An insufficiency of data can bring to consider that a temporal serie is chaotic, while if one had a complete serie, this last it would not be inevitably chaotic. Consequently, a spectral analysis becomes an interesting and important aspect. This last is a tool allowing to analyze properties of frequencies in a system which varies in the time [13].

The spectral study consists to calculate and to draw Fourier transform of the autocorrelation function of x and y. The correspondent spectrums are represented in the figure 6. One can observe easily that the spectrum is continuous and contains many frequential components. A disturbed spectrum like this is characteristic of a chaotic behavior.

5.3 Lyapunov exponents

The determination of the Lyapunov exponents [25] of the dynamical systems is not inevitably easy. One knows that all dynamical system having at least a positive exponent is defined as being chaotic and that the magnitude of this exponent reflects the scale of time on which this system becomes unpredictable. One of the most efficient methods consists to measure the average exponential rate of divergence or convergence of neighbor orbits in a phase space.

For a one dimensional discrete dynamical system $x_{k+1} = f(x_k)$, with initial condition x_0, the Lyapunouv exponent $\lambda(x_0)$ is defined as:

$$\lambda(x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \lg |f'(x_k)| = \lim_{N\to\infty} \frac{1}{N} \lg(\prod_{k=1}^{N} |f'(x_k)|) \tag{16}$$

Of course, the limit in question may be not easy to calculate. In practise, it is often helpful to use a "finite time" approximation to get some idea of what the Lyapunouv exponent may be for a given orbit. Explicitly, for an orbit, the finite time analog of the Lyapunov exponent $\lambda(x_0)$ is :

$$\lambda(x_0, N) = \frac{1}{N} \sum_{k=1}^{N} \lg |f'(x_k)| \tag{17}$$

In this sum the number λ depends on several parameters, notably the synaptic weights. Taking into account the matrix W, one can demonstrate easily that for $a > 1$ the Lyapunov exponent is strictly positive, it is equally possible to find the condition on b, but we know well that an alone positive exponent allows us to deduce the chaotic behaviour in the system (see table I).

5.4 Fractal dimension

The instantaneous state of a dynamic system is characterized by a point in the phase space. A sequence of these states subsequently to the time defines the path in the phase space. If the dynamics of the studied system is reducible to a system of determinist laws, then the system arrives to a permanent state regime. This fact reflect by the convergence of the totality of paths of the phase space. It is the attractor of the system.

Now, we can test the existence of an attractor and evaluate its dimensions [20]. An attractor becomes strange if it has a fractal dimension and therefore, the dynamics system relative to this attractor is chaotic.

There exists several fractal dimensions, we have chosen two dimensions: the box counting dimension and the correlation dimension. This choice is founded on the possible application to a dynamical systems and their implementation is relatively easy.

5.4.1 Box-counting dimension

One obtains it roughly by seeking the number $N(\varepsilon)$, which represents the boxes number of diameter ε which is necessary to cover the set to evaluate. This dimension D_f is given by the following equation:

$$D_f = \lim_{\varepsilon \to 0} \frac{\ln(\frac{1}{N(\varepsilon)})}{\ln(\varepsilon)} \tag{18}$$

Generally, the computation of this dimension is made by counting the number of spheres not empty for each ε. To calculate the limit when $\varepsilon \to 0$, we make an interpolation of a points cloud having as ordinates $\ln(N(\varepsilon))$, corresponding to abscissas $\ln(\varepsilon)$. From its slope, we calculate then the fractal dimension.

This algorithm of box-counting gives good results, but possesses nevertheless two numerical disadvantages:

- The convergence is enough slow and it is necessary to possess a very great number of points.
- To count the number of boxes necessitates a computation enough expensive in time and in space.

5.4.2 Correlation dimension

To evaluate this dimension, we apply the Grassberger-Procaccia method [9]. It consists to reconstruct the space phase and more particularly the points which the coordinates are defined by:

$$X^n(t_i) = [X(t_i), X(t_i + \tau), ..., X(t + (n-1)\tau)]^T \tag{19}$$

A point of reference $x(t_i)$ is chosen and all the distances between this point and the other points are calculated according to the formula:

$$|X(t_i) - X(t_j)| \tag{20}$$

This allows us to calculate the number of present points to a distance r from the chosen point in the phase space. If we repeat this for all the points, (for all values of i) and we calculate the sum, we obtain the correlation integral which its formula is the following:

$$C(r) = \frac{1}{N^2} \sum_{i,j=1}^{N} \phi(r - |X(t_i) - X(t_j)|) \tag{21}$$

Where ϕ is a function of Heaviside with $\phi(x) = 0$ for $x < 0$ and $\phi(x) = 1$ for $x > 0$. In a certain dimension of r, $C(r)$ behaves as if:

$$C(r) = r^d \tag{22}$$

The attractor dimension d can be determined from the computation of the slope $(\ln C(r)/\ln(r))$ [2, 9].

The results of the simulation of these algorithms (Lyapunov exponent, fractal dimension) are recapitulated in the table I.

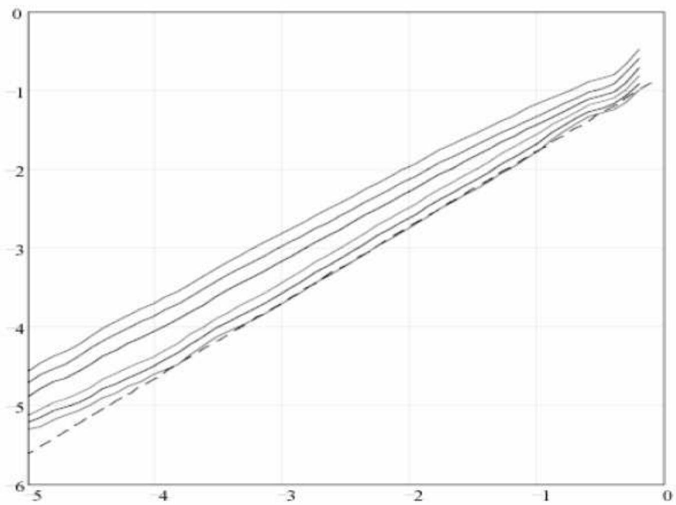

Figure 5: Fractal dimension from the graph $Ln(C(r))$ vs. $Ln(r)$. (With embedding dimensions $m = 1$ through $m = 8$).Equilibrium state.

6 Discussion of results

The different simulations have allowed us to obtain two Lyapunov exponents (See table I), the first is positive and the second is negative, asserting both the chaotic state of system and the attractors divergence in the phase space (See figure 4), thus a fractal dimension close to the value 1, which corresponds to:

- The attractors having a close structure of line curved in the space phase.
- Quasi-periodic evolution in the time space (See figure 3 and 4).

From these verifications, thus of the qualitative and quantitative study undertaken above, we can conclude that the system of neuron which its dynamic follows the equations of the system 1 evolves chaotically.

7 Conclusion

In this work, one has studied dynamic properties of a small discrete neural network. We have determined the equilibrium states as well as their nature of stability by varying the network parameters which are the synaptic weights w_{ij} and the neuron gain σ. In a second time we have seen the system bifurcation which has shown that the system can become chaotic through the period-doubling route as the parameter varies. This has incited us to measure the chaos appearing in this system by computing the Lyapunov exponents and fractal dimension.

In this way one finds great perspectives such us, the application of the chaotic neural network in the different areas (information processing, associative memories, pattern recognition and classification, combinatorial optimization, segmentation and bending of objects, etc.), without forgetting to perform experiments with larger networks.

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