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Analysis of Dual Functions

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Abstract

The purpose of this paper is to develop a theory, inspired from complex analysis, of dual functions. In detail, we introduce the notion of holomorphic dual functions and we establish a general representation of holomorphic dual functions. As an application, we generalize some usual real functions to the dual plane. Finally, we will define the integral trough curves of any dual functions as well as the dual primitive.

Keywords: Dual number; dual function; holomorphicity; integration; primitive.

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1 Introduction

A dual number z is an ordered pair of real numbers (x, y) associated with the real unit 1 and the dual unit ε, where ε is an nilpotent number i.e. $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. A dual number is usually denoted in the form

$$z = x + y\varepsilon. \tag{1}$$

Thus, the dual numbers are elements of the $2-$dimensional real algebra

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \left\{ z = x + y\varepsilon \mid (x, y) \in \mathbb{R}^2, \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0 \right\}, \tag{2}$$

generated by 1 and ε.

There are many manners to choose the dual unit number ε, see for more details and examples the references [1, 5].

Addition and multiplication of the dual numbers are defined by

$$(x_1 + y_1\varepsilon) + (x_2 + y_2\varepsilon) = (x_1 + x_2) + (y_1 + y_2)\,\varepsilon, \quad (3)$$
$$(x_1 + y_1\varepsilon) \,.\, (x_2 + y_2\varepsilon) = (x_1 x_2) + (x_1 y_2 + x_2 y_1)\,\varepsilon. \quad (4)$$

This multiplication is commutative, associative and distributes over addition.

The algebra of dual numbers $\mathbb{D}$ has the numbers εy, $y \in \mathbb{R}$, as divisors of zero. No number εy has an inverse in the algebra $\mathbb{D}$.

One can verify, using (4), that

$$(x + y\varepsilon)^n = x^n + nx^{n-1}y\varepsilon. \quad (5)$$

The conjugate $\bar{z}$ of the dual number $z = x + \varepsilon y$ is defined by

$$\bar{z} = x - y\varepsilon. \quad (6)$$

So

$$z\bar{z} = x^2. \quad (7)$$

The division of two dual numbers can be computed as

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{\overline{z_2}z_2} = \frac{x_1x_2 + (x_1y_2 - x_2y_1)\,\varepsilon}{x_2^2}, \quad (8)$$

Thus, the division $\frac{z_1}{z_2}$ is possible and unambiguous if $x_2 \neq 0$.

Dual numbers can be represented as follows:

- Gaussian representation: $z = x + y\varepsilon$.
- Polar representation: $z = x\,(1 + \varepsilon \arg z)\,,$ where $\arg z = \frac{y}{x}$, $x \neq 0$, is the argument of z.

It follows from this representation that

$$\arg\,(z_1 + z_2) = \arg z_1 + \arg z_1. \quad (9)$$

If $z = x + \varepsilon y$ is a dual number, we denote by real and imaginary (dual) parts of z, the real numbers

$$real\,(z) = x \text{ and } Im\,(z) = y. \quad (10)$$

The theory of algebra of dual numbers has been originally introduced by W. K. Clifford [1] in 1873, and he showed that they form an algebra but not a field because only dual numbers with real part not zero possess an inverse element. In 1891 E. Study [11] realized that this associative algebra was ideal for describing the group of motions of three-dimensional space. At the turn of the 20th century, A. Kotelnikov [6] developed dual vectors and dual quaternions.

Algebraic study of dual numbers are the topic of numerous papers, e.g. [1, 5].

This nice concept has lots of applications in many fields of fundamental sciences; such, algebraic geometry, Riemannian geometry, quantum mechanics and astronomy. It also arises in various contexts of engineering: aerospace, robotic and computer science. For more details about the applications of dual numbers, we refer the reader to [2, 3, 4, 6, 8, 9, 11, 12, 13]

However, up to now there are only a few attempts in the mathematical study of dual functions (functions of dual variable). An early attempt may be due to E. E. Kramer [7] in 1930. Later, in 2011, Z. Ercan and S. Yüce [3] obtained generalized Euler's and De Moivre's formulas for functions with dual Quaternion variable.

In the study of dual functions, some natural questions raise:

- When and under what conditions a dual function is differentiable ?.
- It is possible to generalize some elementary results of complex analysis to dual functions ?.
- How can one extend regularly real functions to dual variable ?.

The purpose of this work is to contribute to the development of dual functions and we will try to answer some questions.

We start by generalizing the notion of holomorphicity to dual functions. To this aim, as in complex analysis, we study the Differentiability of dual functions. The notion of holomorphicity has been introduced and a general representation of holomorphic functions was shown. Moreover, we provide the basic assumptions that allow us to extend holomorphically real functions to the wider dual plane and we ensure that such an extension is meaningful. As an application, we generalize some usual real functions to dual plane.

Further, we also outline the concept of integral along curves of dual functions as well as the primitives of holomorphic dual functions.

2 Holomorphicity of dual functions

We start by giving some topological definitions and properties of the dual plan $\mathbb{D}$.

Let us introduce the mapping

$$\begin{cases} \mathcal{P}:\mathbb{D}\longrightarrow\mathbb{R}_+, \\ \mathcal{P}\left(z\right)=\left|real\left(z\right)\right|. \end{cases} \tag{11}$$

It is easy to check that

$$\begin{cases} z\bar{z}=\mathcal{P}\left(z\right)^2 \quad \forall z\in\mathbb{D}. \\ \mathcal{P}\left(z_1+z_2\right)\leq\mathcal{P}\left(z_1\right)+\mathcal{P}\left(z_2\right) \quad \forall z_1,z_2\in\mathbb{D}, \\ \mathcal{P}\left(z_1z_2\right)=\mathcal{P}\left(z_1\right)\mathcal{P}\left(z_2\right) \quad \forall z_1,z_2\in\mathbb{D}, \\ \mathcal{P}\left(\lambda z\right)=\left|\lambda\right|\mathcal{P}\left(z\right) \quad \forall z\in\mathbb{D},\forall\lambda\in\mathbb{R}, \\ \mathcal{P}\left(0\right)=0. \end{cases} \tag{12}$$

Particularly, $\mathcal{P}$ defines a semi-modulus in $\mathbb{D}$.

Thus, we can define the dual disk and dual circle of centre $z_0 = x_0 + y_0\varepsilon \in \mathbb{D}$ and radius $r > 0$, respectively, by

$$\begin{aligned} D(z_0, r) &= \{z = x + y\varepsilon \in \mathbb{D} \mid p(z - z_0) < r\} \\ &= \{z = x + y\varepsilon \in \mathbb{D} \mid |x - x_0| < r,\ y \in \mathbb{R}\}, \end{aligned} \tag{13}$$

$$\begin{aligned} S(z_0, r) &= \{z = x + y\varepsilon \in \mathbb{D} \mid p(z - z_0) = r\} \\ &= \{z = x + y\varepsilon \in \mathbb{D} \mid |x - x_0| = r,\ y \in \mathbb{R}\}. \end{aligned} \tag{14}$$

$S(z_0, r)$ is also called Galilean circle.

Definition 1 1. We say that Ω is a dual subset of the dual plan $\mathbb{D}$ if there exists a subset $O \subset \mathbb{R}$ such that

$$\Omega = O \times \mathbb{R}. \tag{15}$$

O is called the generator of Ω.

We say that Ω is an open dual subset of the dual plan $\mathbb{D}$ if the generator of Ω is an open subset of $\mathbb{R}$.

2. Ω is said to be a closed dual subset of $\mathbb{D}$ if its complementary is an open subset of $\mathbb{D}$.

3. Ω is said to be a connected dual subset of $\mathbb{D}$ if its generator is a connected subset of $\mathbb{R}$ (real interval).

We discuss now some properties of dual functions. We investigate the continuity of dual functions and the derivability in the dual sense, which can be also called holomorphicity, as in complex case.

To this end, we will need these.

Definition 2 A dual function is a mapping from a subset $\Omega \subset \mathbb{D}$ to $\mathbb{D}$.

Definition 3 A dual function f defined on subset $\Omega \subset \mathbb{D}$ is called homogeneous dual function if

$$f(real(z)) \in \mathbb{R}.$$

In following definitions, we admit that $\mathbb{D}$ is provided by the usual topology of $\mathbb{R}^2$.

Let Ω be an open subset of $\mathbb{D}$, $z_0 = x_0 + y_0\varepsilon \in \Omega$ and $f : \Omega \longrightarrow \mathbb{D}$ a dual function.

Definition 4 We say that the function f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{16}$$

Definition 5 The function is continuous in $\Omega \subset \mathbb{D}$ if it is continuous at every point of Ω.

Definition 6 The dual function f is said to be differentiable at $z_0 = x_0 + y_0\varepsilon$ (in the dual sense), if the limit below exists

$$\frac{df}{dz}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{17}$$

$\frac{df}{dz}(z_0)$ is called the derivative of f at the point z_0.

If f is differentiable for all points in a neighborhood of the point z_0 then f is called holomorphic at z_0.

Definition 7 The function f is holomorphic in $\Omega \subset \mathbb{D}$ if it is holomorphic at every point of Ω.

The definition of derivative in the dual sense has to be treated with a little more care than its real companion; this is illustrated by the following example.

Example 1 The function $f(z) = \overline{z}$ is nowhere differentiable. To this aim, a simple calculation gives

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\ &= \lim_{z\to z_0} \frac{\overline{z - z_0}^2}{(x - x_0)^2} \\ &= 1 - 2\varepsilon \lim_{x\to x_0, y\to y_0} \frac{y - y_0}{x - x_0}.\end{aligned}$$

But this limit does not exist.

The basic properties for derivatives are similar to those we know from real calculus. In fact, one should convince oneself that the following rules follow mostly from properties of the limit.

Lemma 1 Suppose f and g are differentiable at $z \in \mathbb{D}$, and that $c \in \mathbb{D}$, $n \in \mathbb{Z}$, and h is differentiable at $g(z)$.

1. $\frac{d(f+cg)}{dz} = \frac{df}{dz} + c\frac{dg}{dz}$.
2. $\frac{d(f.g)}{dz} = \frac{df}{dz}g + f\frac{dg}{dz}$.
3. $\frac{d\left(\frac{f}{g}\right)}{dz} = \frac{\frac{df}{dz}g - f\frac{dg}{dz}}{g^2}$ (we have to be aware of division by zero).
4. $\frac{dh\circ g}{dz} = \frac{dh}{dz}(g)\frac{dg}{dz}$.

In the following results we generalize the Cauchy-Riemann formulas to dual functions.

Theorem 2 Let f be a dual function in $\Omega \subset \mathbb{D}$, which can be written in terms of its real and dual parts as

$$f = \varphi + \varepsilon\psi.$$

f is holomorphic in $\Omega \subset \mathbb{D}$ if and only if the derivative of f satisfies

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial x}\varepsilon. \tag{18}$$

Proof. Suppose that f is holomorphic in Ω and let $z_0 = x_0 + \varepsilon y_0$ be an arbitrary element of Ω.

By definition of derivative, we have

$$\begin{aligned}\frac{df}{dz}(z_0) &= \lim_{x\to x_0} \frac{f(x + y\varepsilon) - f(x_0 + y_0\varepsilon)}{(x + y\varepsilon) - (x_0 + y_0\varepsilon)} \\ &= \lim_{x\to x_0, y\to y_0} \frac{\varphi(x,y) - \varphi(x_0,y_0)}{x - x_0} + \lim_{x\to x_0, y\to y_0} \frac{\psi(x,y) - \psi(x_0,y_0)}{x - x_0}\varepsilon + \\ &\quad \lim_{x\to x_0, y\to y_0} \frac{(\varphi(x,y) - \varphi(x_0,y_0))(y - y_0)}{(x - x_0)^2}\varepsilon. \\ &= \frac{\partial \varphi}{\partial x}(x_0,y_0) + \frac{\partial \psi}{\partial x}(x_0,y_0)\varepsilon + \\ &\quad \lim_{x\to x_0, y\to y_0} \frac{(\varphi(x,y) - \varphi(x_0,y_0))(y - y_0)}{(x - x_0)^2}\varepsilon.\end{aligned}$$

Then, the limit exists if and only if it does not depend on limit of the bounded ratio $\frac{y-y_0}{x-x_0}$. Hence, we have two cases to deal with

First case. Let us remark that

$$\lim_{x\to x_0,y\to y_0}\frac{\left(\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)\right)\left(y-y_0\right)}{\left(x-x_0\right)^2}=\lim_{x\to x_0,y\to y_0}\frac{\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)}{x-x_0}\frac{y-y_0}{x-x_0}.$$

Therefore, the above limit exists if

$$\lim_{x\to x_0,y\to y_0}\frac{\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)}{x-x_0}=\frac{\partial\varphi}{\partial x}\left(x_0,y_0\right)=0. \tag{19}$$

This implies that

$$\frac{df}{dz}\left(z_0\right)=\frac{\partial\psi}{\partial x}\left(x_0,y_0\right)\varepsilon.$$

Which gives (18).

Second case. We can also write

$$\lim_{x\to x_0,y\to y_0}\frac{\left(\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)\right)\left(y-y_0\right)}{\left(x-x_0\right)^2}=\lim_{x\to x_0,y\to y_0}\frac{\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)}{y-y_0}\left(\frac{y-y_0}{x-x_0}\right)^2.$$

Hence, the limit exists if

$$\lim_{x\to x_0,y\to y_0}\frac{\varphi\left(x,y\right)-\varphi\left(x_0,y_0\right)}{x-x_0}=\frac{\partial\varphi}{\partial y}\left(x_0,y_0\right)=0. \tag{20}$$

Clearly, (18) follows. This permits us to conclude the proof.

Corollary 3 Let f be a dual function in $\Omega\subset\mathbb{D}$, which can be written in terms of its real and dual parts as $f=\varphi+\varepsilon\psi$ and suppose that the partial derivatives of f exist. Then,

1. f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if its partial derivatives satisfy

$$\varepsilon\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}. \tag{21}$$

2. f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if the following formula holds

$$\left\{\begin{array}{l}\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y},\\ \frac{\partial\varphi}{\partial y}=0.\end{array}\right. . \tag{22}$$

Proof. 1. In view of (18), we can assert that the total differential of f can be written

$$\begin{aligned}df&=\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial x}\varepsilon\right)d\left(x+\varepsilon y\right)\\&=\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial x}\varepsilon\right)dx+\frac{\partial\varphi}{\partial x}\varepsilon dy.\end{aligned} \tag{23}$$

This yields,

$$\left\{\begin{array}{l}\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial x}\varepsilon,\\ \frac{\partial f}{\partial y}=\frac{\partial\varphi}{\partial x}\varepsilon,\end{array}\right. \tag{24}$$

and so

$$\varepsilon\frac{\partial f}{\partial x}=\varepsilon\frac{\partial \varphi}{\partial x}.$$

Which eventually, combined with the second equation in (24), gives (21).

2. The formula (21) leads to

$$\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \psi}{\partial x}\varepsilon\right)\varepsilon=\frac{\partial \varphi}{\partial y}+\frac{\partial \psi}{\partial y}\varepsilon.$$

Hence, (22) results.

Theorem 4 The function f is holomorphic in the open subset $\Omega\subset\mathbb{D}$, (with respect to the topology of $\mathbb{R}^2$), if and only if there exists a pair of real functions φ and k, such that $\varphi\in C^1\left(P_x\left(\Omega\right)\right)$, $\frac{d\varphi}{dx}$ is differentiable in $P_x\left(\Omega\right)$ and k is differentiable in $P_x\left(\Omega\right)$, where P_x is the first projection (parallel to the vertical axis), so that the function f can be written explicitly

$$f\left(z\right)=\varphi\left(x\right)+\left(\frac{d\varphi}{dx}y+k\left(x\right)\right)\varepsilon\quad\forall z\in\Omega. \tag{25}$$

Proof. Since f is holomorphic in Ω, we find, employing (22)

$$\begin{cases}\frac{\partial \varphi}{\partial x}=\frac{\partial \psi}{\partial y},\\ \frac{\partial \varphi}{\partial y}=0.\end{cases}$$

It follows that

$$\varphi\left(x,y\right)=\varphi\left(x\right).$$

Hence

$$\frac{\partial \psi}{\partial y}=\frac{d\varphi}{dx}.$$

So, we find

$$\psi\left(x,y\right)=\frac{d\varphi}{dx}y+k\left(x\right).$$

This achieves the proof.

Remark 1 1. The formula (25) gives, taking into account the fact that $\frac{df}{dz}=\frac{\partial f}{\partial x}$,

$$\frac{df}{dz}=\frac{d\varphi}{dx}+\left(\frac{d^2\varphi}{dx^2}y+\frac{dk}{dx}\right)\varepsilon. \tag{26}$$

In addition, if the functions φ and k are enough regular, we can generalize this relation by recurrence.

2. If, in particular, f is an homogeneous function, formula (2.15) gives $k\equiv 0$.

3. We remark that f is linear with respect to the variable y. Then, f can be holomorphically extended to the dual subset $P_x\left(\Omega\right)\times\mathbb{R}$.

The lemma below can be easily deduced.

Lemma 5 Let f be an homogeneous holomorphic function in the open dual subset $O\times\mathbb{R}\subset\mathbb{D}$. Then

$$\overline{f\left(z\right)}=f\left(\overline{z}\right)\quad\forall z\in O\times\mathbb{R}. \tag{27}$$

We are interested now to some properties regarding constant dual functions. To this aim, we need the following definition.

Definition 8 Let f be a dual function in the subset $\Omega \subset \mathbb{D}$.

1. f is said to be bounded in Ω if there exists a positive constant c such that

$$\|f(z)\|_{\mathbb{R}^2} \leq c \ \ \forall z \in \Omega, \tag{28}$$

which means that

$$\exists c > 0 \mid (realf(z))^2 + (Imf(z))^2 \leq c \ \ \forall z \in \Omega. \tag{29}$$

2. f is said to be bounded in the dual sense in Ω if there exists a positive constant c such that

$$\mathcal{P}(z) \leq c \ \ \forall z \in \Omega. \tag{30}$$

Proposition 6 1. Let f be an holomorphic function in the open dual subset $O \times \mathbb{R} \subset \mathbb{D}$. If there exists a connected and compact subset K, (bounded interval), contained in O such that f is bounded in K. Then, there exists a differentiable real function k and a constant $C(K)$, depending on K, such that the following formula holds

$$f(z) = C(K) + k(x)\varepsilon \ \ \forall z \in K \times \mathbb{R}. \tag{31}$$

2. Let f be an homogeneous holomorphic function in the open connected dual subset $O \times \mathbb{R} \subset \mathbb{D}$. If f is bounded in $O \times \mathbb{R}$, then it is necessary constant in $O \times \mathbb{R}$.

Proof. 1. Making use (25), we have

$$f(z) = \varphi(x) + \left(\frac{d\varphi}{dx} y + k(x)\right)\varepsilon \ \ \forall z \in O \times \mathbb{R},$$

whence $\varphi \in C^1(O)$, $\frac{d\varphi}{dx}$ is differentiable in O and k is differentiable in O.

Suppose that there exists a compact K contained in O such that f is bounded in K, then there exists $c > 0$ verifying, for all $x \in K$ and $y \in \mathbb{R}$,

$$\begin{aligned} |\varphi(x)| &\leq c, \\ \left|\left(\frac{d\varphi}{dx} y + k(x)\right)\right| &\leq c. \end{aligned}$$

Thus, we can infer that φ is bounded in O and

$$\left|\frac{d\varphi}{dx}\right| |y| \leq c + |k(x)| \ \ \forall x \in K, \ \forall y \in \mathbb{R}.$$

Which asserts us, keeping in mind that k is continuous in K, that

$$\left|\frac{d\varphi}{dx}\right| |y| \leq c + \max_K |k(x)| \ \ \forall x \in K, \ \forall y \in \mathbb{R}.$$

Consequently, since O is connected, we deduce that φ is constant.

2. Let f be an homogeneous holomorphic function in an open connected dual subset $O\times\mathbb{R}\subset\mathbb{D}$. Then, by definition, we have $k\equiv 0$.

Hence, since $O\times\mathbb{R}$ is supposed to be connected, the result follows, by proceeding as in the proof of the first assertion.

Proposition 7 If an holomorphic function f is defined in an open connected dual subset $O\times\mathbb{R}\subset\mathbb{D}$ and $\frac{df}{dz}=0$ for all z in $O\times\mathbb{R}$, then f is constant.

Proof. In view of Theorem 4, there exists a pair of real functions φ and k, where $\varphi\in C^1(O)$, $\frac{d\varphi}{dx}$ is differentiable in O and k is differentiable in O, such that

$$f(z)=\varphi(x)+\left(\frac{d\varphi}{dx}y+k(x)\right)\varepsilon\ \ \forall x\in O.$$

Under this formula the derivative of f can be written

$$\frac{df}{dz}=\frac{d\varphi}{dx}+\left(\frac{d^2\varphi}{dx^2}y+\frac{dk}{dx}\right)\varepsilon.$$

From which, we can infer

$$\begin{aligned}\frac{d\varphi}{dx}&=0\ \ \forall x\in O,\\ \frac{d^2\varphi}{dx^2}y+\frac{dk}{dx}&=0\ \ \forall x\in O,\ \forall y\in\mathbb{R}.\end{aligned}$$

Then, since O is connected, we conclude that φ and k are constant.

The following Proposition shows that we can extend any regular real function to the dual plane $\mathbb{D}$.

Proposition 8 Let O an open subset of $\mathbb{R}$, $f\in\mathcal{C}^1(O)$ and $\frac{df}{dx}$ is differentiable in O. Then, there exists a unique homogeneous holomorphic function $F:O\times\mathbb{R}\subset\mathbb{D}\longrightarrow\mathbb{D}$ such that

$$F(x)=f(x)\ \ \forall x\in O. \tag{32}$$

Moreover, if $f\in\mathcal{C}^k(O)$ then $F\in\mathcal{C}^{k-1}(O\times\mathbb{R})$.

Proof. Considering the dual function F given by

$$F(z)=F(x+\varepsilon y)=f(x)+\frac{df}{dx}y\varepsilon\ \ \forall z\in O\times\mathbb{R}.$$

It is clear that F is an homogeneous holomorphic function in $O\times\mathbb{R}$ and verifies

$$F(x)=f(x)\ \ \forall x\in O.$$

Suppose that there exists another holomorphic function g in $O\times\mathbb{R}$ such that

$$g(x)=f(x)\ \ \forall x\in O.$$

It is well known that there exists a real function φ, where $\varphi\in C^1(O)$ and $\frac{d\varphi}{dx}$ is differentiable in O, satisfying

$$g(z)=\varphi(x)+\frac{d\varphi}{dx}y\varepsilon.$$

Hence,

$$g(x) = \varphi(x) \quad \forall x \in O,$$

and so

$$\varphi(x) = f(x) \quad \forall x \in O.$$

The second assertion is an immediate consequence.

3 Usual dual functions

We can think of applying the statement of proposition 8, which asserts that any regular real function on its domain can be holomorphically extended to dual numbers, to build homogeneous dual functions similar to the usual real functions, obtained as their extensions.

3.1 The dual exponential function

The real exponential function e^x defined for all $x \in \mathbb{R}$ can be extended to the dual plane $\mathbb{D}$ as follows

$$\exp(z) = e^z = e^x + e^x y\varepsilon = e^x (1 + y\varepsilon). \tag{33}$$

The derivative of e^z is

$$\frac{de^z}{dz} = \frac{de^x}{dx} + \frac{de^x}{dx} y\varepsilon = e^z \quad \forall z \in \mathbb{D}. \tag{34}$$

By recurrence, we find

$$\frac{d^n e^z}{dz^n} = e^z \quad \forall z \in \mathbb{D}, \quad \forall n \in \mathbb{N}. \tag{35}$$

Thus, any dual number $z = x + y\varepsilon \in \mathbb{D}$, $x \neq 0$ has the exponential representation

$$z = xe^{(\arg z)\varepsilon}. \tag{36}$$

Some properties of the dual exponential function are collected in the following.

Proposition 9 1. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.
2. $e^{-z} = \frac{1}{e^z}$.
3. $e^z \neq 0 \quad \forall z \in \mathbb{D}$.
4. $\exp \in C^\infty(\mathbb{D})$.

3.2 The dual trigonometric functions

The trigonometric functions: sine, cos, tangent, etc, have their dual analogues. In fact, we can define them by the formulas

$$\sin z = \sin x + (\cos x) y\varepsilon \quad \forall z \in \mathbb{D}, \tag{37}$$

$$\cos z = \cos x - (\sin x) y\varepsilon \ \forall z \in \mathbb{D}, \tag{38}$$

$$\tan z = \tan x - \frac{y}{\cos^2 x}\varepsilon = \frac{\sin z}{\cos z} \ \forall z \in \mathbb{D} - \{(2k+1)\pi,\ k \in \mathbb{Z}\} \times \mathbb{R}. \tag{39}$$

The next properties follow mostly from the previous definition.

Proposition 10 1. sin, cos and tan are $2\pi-$periodic functions.
2. $\sin(-z) = -\sin z,\ \ \cos(-z) = \cos z$.
3. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
4. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
5. $\sin^2 z + \cos^2 z = 1$.
6. $\cos(\varepsilon x) = 1,\ \ \sin(\varepsilon x) = \varepsilon x$.
7. $\frac{d \sin z}{dz} = \cos z$.
8. $\frac{d \cos z}{dz} = -\sin z$.
9. sin, cos $\in C^\infty(\mathbb{D})$ and $\tan \in C^\infty(\mathbb{D} - \{(2k+1)\pi,\ k \in \mathbb{Z}\} \times \mathbb{R})$.

3.3 The dual hyperbolic functions

The dual hyperbolic functions are defined by

$$\sinh z = \sinh x + (\cosh x)\, y\varepsilon \ \ \forall z \in \mathbb{D}, \tag{40}$$

$$\cosh z = \cosh x + (\sinh x)\, y\varepsilon \ \forall z \in \mathbb{D}, \tag{41}$$

$$\tan z = \frac{\sinh z}{\cosh x} \ \forall z \in \mathbb{D}. \tag{42}$$

These are equivalent, as in the real case, to

$$\sinh z = \frac{e^z - e^{-z}}{2} \ \ \forall z \in \mathbb{D}, \tag{43}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \ \forall z \in \mathbb{D}, \tag{44}$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} \ \forall z \in \mathbb{D}. \tag{45}$$

The following collects some basic properties.

Proposition 11 1. $\sinh(-z) = -\sinh z,\ \ \cosh(-z) = \cosh z$.
2. $\cosh^2 z - \sinh^2 z = 1$.
3. $\cosh(\varepsilon x) = 1,\ \ \sinh(\varepsilon x) = \varepsilon x$.
4. $\frac{d \sinh z}{dz} = \cosh z$.
5. $\frac{d \cosh z}{dz} = \sinh z$.
6. sinh, cosh, tanh $\in C^\infty(\mathbb{D})$.

3.4 The dual logarithmic function

We define the dual Logarithmic function by the formula

$$\log z = \log x + \frac{y}{x}\varepsilon = \log x + (\arg z)\,\varepsilon \ \ \forall z \in \mathbb{R}_+^* \times \mathbb{R} \subset \mathbb{D}. \tag{46}$$

The dual Logarithmic function, satisfies some properties, given by

Proposition 12 1. $\log\left(\frac{1}{z}\right) = -\log z$.
2. $\log(z_1 z_2) = \log z_1 + \log(z_2)$.
3. $e^{\log z} = \log(e^z) = z$.
4. $\log(z^\alpha) = \alpha \log z \quad \forall z \in \mathbb{R}_+^* \times \mathbb{R}, \forall \alpha \in \mathbb{R}$.
5. $\frac{d\log z}{dz} = \frac{1}{z}$.
6. $\log \in C^\infty\left(\mathbb{R}_+^* \times \mathbb{R}\right)$.

Example 2 We can evaluate the quantity z^w for all $z = x + y\varepsilon \in \mathbb{R}_+^* \times \mathbb{R}$ and $w = a + b\varepsilon \in \mathbb{D}$ as follows

$$\begin{aligned} z^w &= e^{(a+b\varepsilon)\left(\log x + \frac{y}{x}\varepsilon\right)} \\ &= e^{a\log x + \left(a\frac{y}{x} + b\log x\right)\varepsilon} \\ &= e^{a\log x}\left(1 + \left(a\frac{y}{x} + b\log x\right)\varepsilon\right). \end{aligned}$$

4 Integration of dual functions

We begin integration by focusing on "1−dimensional" integrals over lines.

At first sight, dual integration is not really anything different from real integration. For a continuous dual-valued function $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{D}$, we define

$$\int_a^b f(t)\,dt = \int_a^b real\,(f(t))\,dt + \varepsilon \int_a^b Im\,(f(t))\,dt. \tag{47}$$

For a function which takes dual numbers as arguments, we integrate along a curve γ (instead of a real interval). Suppose this curve is parameterized by $\gamma(t)$, $a \le t \le b$. The following definition is used.

Definition 9 Suppose γ is a smooth curve parameterized by $\gamma(t)$, $a \le t \le b$, and f is a continuous dual function on γ. Then, we define the integral of f on γ as

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\frac{d\gamma}{dt}dt. \tag{48}$$

This definition can be easily and naturally extended to piecewise smooth curves. In what follows, we will usually state our results for smooth curves, bearing in mind that practically all can be extended to piecewise smooth curves.

The dual integral has some standard properties, most of which follow from real case. The first property to observe is that the actual choice of parameterization of γ does not matter.

Proposition 13 Let γ be a smooth curve and let f a continuous dual function on γ. Then, the integral $\int_\gamma f(z)\,dz$ is independent of the parameterization of γ chosen.

We give now the following Proposition, which is a direct consequence of the definition of dual integral.

Proposition 14 1. Suppose γ is a smooth curve, f and g are dual functions which are continuous on γ and let $c \in \mathbb{D}$. Then

$$\int_\gamma (cf(z) + g(z))\,dz = c\int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz. \tag{49}$$

2. If γ is parameterized by $\gamma(t)$, $a \leq t \leq b$, define the curve $-\gamma$ by $-\gamma(t) = \gamma(a+b-t)$. Then

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz. \tag{50}$$

3. If γ_1 and γ_2 are smooth curves so that γ_2 starts where γ_1 ends then define the curve $\gamma_1 \circ \gamma_2$ by following γ_1 to its end, and then continuing on γ_2 to its end. Then

$$\int_{\gamma_1 \circ \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz. \tag{51}$$

4. Suppose that γ is a smooth curve parallel to the vertical axis, i.e. γ is parameterized by $\gamma(t) = (\alpha, y(t))$, $a \leq t \leq b$ and $\alpha \in \mathbb{R}$, then

$$real\left(\int_{\gamma} f(z)\,dz\right) = 0, \quad \text{(pure dual number)}. \tag{52}$$

Now, we claim and prove the following Theorem, which generalizes that of Cauchy-Goursat to dual functions.

Theorem 15 Let O be an open connected subset of $\mathbb{R}$ and let f be an holomorphic function in the dual subset $O \times \mathbb{R} \subset \mathbb{D}$. Then, the integral of f vanishes along any rectangle contained in $O \times \mathbb{R}$.

Proof. To simplify the proof we restrict ourselves to the rectangle $R = \{z_1, z_2, z_3, z_4\}$, whence $z_1 = x_1 + \varepsilon y_1$, $z_2 = x_2 + \varepsilon y_1$, $z_3 = x_2 + \varepsilon y_2$, $z_4 = x_1 + \varepsilon y_2$, such that $a \leq x_1 \leq x_2 \leq b$ and $y_1 \leq y_2$.

As already known, there exists a pair of real functions φ and k, such that $\varphi \in C^1(O)$, $\frac{d\varphi}{dx}$ is differentiable in O and k is differentiable in O, satisfying

$$f(z) = \varphi(x) + \left(\frac{d\varphi}{dx} y + k(x)\right)\varepsilon \quad \forall z \in O \times \mathbb{R}.$$

Then, the integral of f becomes

$$\begin{aligned}
\int_{\partial R} f(z)\,dz &= \int_{\partial R} \left[\varphi(x) + \left(\frac{d\varphi}{dx} y + k(x)\right)\varepsilon\right](dx + \varepsilon dy) \\
&= \int_{\partial R} \left[\varphi(x) + \left(\frac{d\varphi}{dx} y + k(x)\right)\varepsilon\right] dx + \varepsilon \varphi(x)\,dy \\
&= \int_{x_1}^{x_2} \left[\varphi(x) + \left(\frac{d\varphi}{dx} y_1 + k(x)\right)\varepsilon\right] dx + \varepsilon \int_{y_1}^{y_2} \varphi(x_2)\,dy \\
&\quad - \int_{x_1}^{x_2} \left[\varphi(x) + \left(\frac{d\varphi}{dx} y_2 + k(x)\right)\varepsilon\right] dx - \varepsilon \int_{y_1}^{y_2} \varphi(x_1)\,dy \\
&= \varepsilon(\varphi(x_2) - \varphi(x_1))\, y_1 + \varepsilon(y_2 - y_1)\,\varphi(x_2) - \varepsilon(\varphi(x_2) - \varphi(x_1))\, y_2 \\
&\quad - \varepsilon(y_2 - y_1)\,\varphi(x_1).
\end{aligned}$$

Thus, $\int_{\partial R} f(z)\,dz = 0.$

Theorem 16 Let γ_1 and γ_2 be two homotopic smooth curves parameterized, respectively, by $\gamma_1(t)$ and $\gamma_2(t)$, $0 \leq t \leq 1$ such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ and H denotes a regular homotopy between γ_1 and γ_2 defined by

$$\begin{cases} H : [0,1]^2 \longrightarrow \mathbb{R}, \\ H(t,0) = \gamma_1(t) \quad \forall t \in [0,1], \\ H(t,0) = \gamma_2(t) \quad \forall t \in [0,1], \\ H(0,s) = \gamma_1(0) = \gamma_2(0) \quad \forall s \in [0,1], \\ H(1,s) = \gamma_1(1) = \gamma_2(1) \quad \forall s \in [0,1]. \end{cases} \tag{53}$$

Let f be an holomorphic function in a dual subset $O \times \mathbb{R}$ containing the curves γ_1 and γ_2. If H has continuous second partial derivatives, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz. \tag{54}$$

Proof. Let γ_s be the curve parameterized by $H(t,s)$, $0 \leq t \leq 1$. Consider the function $I(s) = \int_{\gamma_s} f(z)\,dz$.

We will show that I is constant with respect to s. To do this, considering the derivative of I. By some calculations, we obtain

$$\begin{aligned} \frac{dI}{ds} &= \frac{d}{ds}\int_0^1 f(H(t,s))\frac{\partial H}{\partial t}dt \\ &= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial s}\frac{\partial H}{\partial t} + f(H(t,s))\frac{\partial^2 H}{\partial s \partial t} \right) dt \end{aligned}$$

If we assume assumption that H has continuous second partial derivatives, the following equality holds

$$\begin{aligned} \frac{dI}{ds} &= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial t}\frac{\partial H}{\partial s} + f(H(t,s))\frac{\partial^2 H}{\partial t \partial s} \right) dt \\ &= \int_0^1 \frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H}{\partial s} \right) dt. \end{aligned}$$

Then

$$\frac{dI}{ds} = f(H(1,s))\frac{\partial H}{\partial s}(1,s) - f(H(0,s))\frac{\partial H}{\partial s}(0,s) = 0.$$

This obviously completes the proof of the Theorem.

Corollary 17 Let f be an holomorphic function in a connected dual subset $O \times \mathbb{R}$ and $\gamma \subset O \times \mathbb{R}$ be a closed smooth curve. Then $\int_\gamma f(z)\,dz = 0$.

Proof. It is enough to remark that $O \times \mathbb{R}$ is simply connected (with respect to the topology of $\mathbb{R}^2$). Hence, the curve γ is contractible, and so Theorem 16 affirms us that $\int_\gamma f(z)\,dz = 0$.

We introduce now the concept of primitive of dual functions.

Definition 10 Let $O\times\mathbb{R}$ be an open subset of $\mathbb{D}$. For any functions $f, F : O\times\mathbb{R} \longrightarrow \mathbb{D}$, if F is holomorphic in $O \times \mathbb{R}$ and $\frac{dF}{dz} = f(z)$ for all $z \in O \times \mathbb{R}$, then F is a primitive of f in $O \times \mathbb{R}$.

Theorem 18 Every holomorphic function in an open connected subset $O \times \mathbb{R}$ of $\mathbb{D}$ has a primitive.

Proof. Let $f : O \times \mathbb{R} \longrightarrow \mathbb{D}$ be an holomorphic function in $O \times \mathbb{R}$.

Fix $z_0 = x_0 + y_0\varepsilon \in Int\,(O \times \mathbb{R})$ and consider the dual function F defined by

$$F(z) = \int_0^1 (z - z_0)\, f(z_0 + t(z - z_0))\, dt \quad \forall z \in O \times \mathbb{R}.$$

Thus, F possesses partial derivatives with respect to the variables x and y, given by

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \int_0^1 \left(f(z_0 + t(z - z_0)) + (z - z_0)\, t\frac{df}{dz}(z_0 + t(z - z_0)) \right) dt, \\
\frac{\partial F}{\partial y} &= \varepsilon \int_0^1 \left(f(z_0 + t(z - z_0)) + (z - z_0)\, t\frac{df}{dz}(z_0 + t(z - z_0)) \right) dt.
\end{aligned}$$

We can infer

$$\frac{\partial F}{\partial y} = \varepsilon \frac{\partial F}{\partial x}.$$

Which implies that F is holomorphic in $O \times \mathbb{R}$ and its derivative is

$$\begin{aligned}
\frac{dF}{dz} &= \frac{\partial F}{\partial x} = \int_0^1 \left(f(z_0 + t(z - z_0)) + (z - z_0)\, t\frac{df}{dz}(z_0 + t(z - z_0)) \right) dt \\
&= \int_0^1 \frac{d}{dt}\left(t f(z_0 + t(z - z_0)) \right) dt = f(z).
\end{aligned}$$

We conclude that F is a primitive of f.

Theorem 19 Suppose f is continuous in an open connected dual subset of $O\times\mathbb{R} \subset \mathbb{D}$ and

$$\int_\gamma f(z)\, dz = 0, \tag{55}$$

for all smooth closed paths $\gamma \subset O \times \mathbb{R}$. Then f has a primitive in $O \times \mathbb{R}$.

Proof. Fixing a point $z_0 = x_0 + y_0\varepsilon \in Int\,(O \times \mathbb{R})$. For each point $z \in O \times \mathbb{R}$, let γ_z be a smooth curve in $O \times \mathbb{R}$ from z_0 to z.

Introducing the function

$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

We obtain the following both equalities (56) and (57), by integrating along, respectively, the two closed paths below and using the hypothesis (55),

$$\left\{[x_0, x] \times \{y_0\varepsilon\} \cup \{x\} \times [y_0\varepsilon, y\varepsilon] \cup -\gamma_z\right\},$$
$$\left\{\{x_0\} \times [y_0\varepsilon, y\varepsilon] \cup [x_0, x] \times \{y\varepsilon\} \cup -\gamma_z\right\},$$

$$\int_{x_0}^{x} f(s + y_0\varepsilon)\, ds + \int_{y_0}^{y} f(x + t\varepsilon)\, \varepsilon dt - F(z) = 0, \tag{56}$$

$$\int_{y_0}^{y} f(x_0 + t\varepsilon)\, \varepsilon dt + \int_{x_0}^{x} f(s + y\varepsilon)\, ds - F(z) = 0. \tag{57}$$

Owing (56) and (57), we deduce that F possesses partial derivatives with respect the variables x and y, given by

$$\frac{\partial F}{\partial x} = f(z) \quad \text{and} \quad \frac{\partial F}{\partial y} = \varepsilon f(z).$$

This yields

$$\varepsilon \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}.$$

Which implies, making use (21), that F is holomorphic in $O \times \mathbb{R}$ and its derivative is

$$\frac{dF}{dz} = \frac{\partial F}{\partial x} = f(z).$$

We establish here a general representation of primitives of any holomorphic function.

Proposition 20 Let $O \times \mathbb{R}$ be an open connected subset of $\mathbb{D}$. Let $f : O \times \mathbb{R} \longrightarrow \mathbb{D}$ be an holomorphic function in $O \times \mathbb{R}$ given by the representation

$$f(z) = \varphi(x) + \left(\frac{d\varphi}{dx} y + k(x)\right) \varepsilon \quad \forall z \in O \times \mathbb{R}. \tag{58}$$

Then, the primitive of f can be calculated via the following formula

$$F(z) = \int f(z)\, dz = \int \varphi(x)\, dx + \left(\varphi(x)\, y + \int k(x)\, dx\right) \varepsilon + c \quad \forall z \in O \times \mathbb{R}. \tag{59}$$

where c is an arbitrary dual constant.

Proof. Let F be primitive of f, i.e. $\frac{dF}{dz} = f$.

It is well known that there exists a pair of real functions φ and k, such that $\varphi \in C^1(O)$, $\frac{d\varphi}{dx}$ is differentiable in O and k is differentiable in O, satisfying

$$F(z) = \psi(x) + \left(\frac{d\psi}{dx} y + r(x)\right) \varepsilon \quad \forall z \in O \times \mathbb{R}.$$

So

$$f(z) = \frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{d\psi}{dx} + \left(\frac{d^2\psi}{dx^2} y + \frac{dr}{dx}\right) \varepsilon \quad \forall z \in O \times \mathbb{R}.$$

Which implies, employing (58)

$$\begin{aligned} \frac{d\psi}{dx} &= \varphi(x) \quad \forall x \in O, \\ \frac{d^2\psi}{dx^2}y + \frac{dr}{dx} &= \frac{d\varphi}{dx}y + k(x) \quad \forall x \in O,\ \forall y \in \mathbb{R}. \end{aligned}$$

Hence, for every $x \in O$

$$\begin{cases} \psi(x) = \int \varphi(x)\,dx + c_1, \\ r(x) = \int k(x)\,dx + c_2, \end{cases}$$

where c_1 and c_2 are two real constants.

Thus, we obtain

$$\begin{aligned} F(z) &= \left(\int \varphi(x)\,dx + c_1\right) + \left(\varphi y + \int k(x)\,dx + c_2\right)\varepsilon \\ &= \int \varphi(x)\,dx + \left(\varphi(x)\,y + \int k(x)\,dx\right)\varepsilon + (c_1 + c_2\varepsilon) \quad \forall z \in O \times \mathbb{R}. \end{aligned}$$

This completes the proof of the Theorem.

We are now ready to state the following Theorem, showing that the integral of holomorphic functions along any path dependent only on the extremities.

Theorem 21 Suppose $O \times \mathbb{R}$ is an open connected dual subset. Let $f : O \times \mathbb{R} \longrightarrow \mathbb{D}$ be an holomorphic function in $O \times \mathbb{R}$, $\gamma \subset O \times \mathbb{R}$ be a smooth curve with parameterization $\gamma(t)$, $a \leq t \leq b$. If F is any primitive of f in $O \times \mathbb{R}$ then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)). \tag{60}$$

Proof. By definition of the integral along the curve γ, one can check that

$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_\gamma \frac{dF}{dz}dz \\ &= \int_a^b \frac{dF}{dz}(\gamma(t))\frac{d\gamma}{dt}dt \\ &= \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

References

[1] W. K. Clifford. Preliminary sketch of bi-quaternions, Proc. London Math. Soc. 4, 381-395, 1873.

[2] A. C. Çöken, A. Görgülü, On the dual Darboux rotation axis of the dual space curve, Demonstratio Math Vol. 35, No. 2, 385–390, 2002.

[3] Z. Ercan, S. Yüce, On properties of the dual quaternions, Eur. J. Pure Appl. Math, Vol. 4, No. 2, 142-146, 2011.

[4] J. A. Fike, J. J. Alonso, The development of hyper-dual numbers for exact second-derivative calculations, 49th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition, Orlando, Florida, 4-7 January 2011.

[5] W. B. V. Kandasamy, F. Smarandache, Dual numbers, ZIP Publishing, Ohio, 2012.

[6] A. P. Kotelnikov, Screw calculus and some applications to geometry and mechanics, Annal. Imp. Univ. Kazan, 1895. (Russian).

[7] E. E. Kramer, Polygenic functions of the dual variable $w = u + jv$, Amer. J. Math. Vol. 52, No. 2, pp. 370-376, Apr 1930.

[8] E. Pennestrì, R. Stefanelli, Linear algebra and numerical algorithms using dual numbers, Multibody Syst. Dyn. 18.3. 323-344, 2007.

[9] C. Soulé, Rational $K-$theory of the dual numbers of a ring of algebraic integers, Springer Lec. Notes 854, 402-408, 1981.

[10] V. Smirnov, Cours de Mathématiques supérieures - Tome 3, Mir. 1989.

[11] E. Study, Geometrie der dynamen, Teubner, Leipzig, 1901.

[12] G. R. Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, Mechanisms and Machine Theory, vol. 11. pp. 141-156, 1976.

[13] A. T. Yang, F. Freudenstein, Application of dual number quaternion algebra to the analysis of spatial mechanisms, Trans. ASME J. Appl. Mech, vol. 31, no.2, pp. 300-308, 1964.