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Dynamic Behaviour of a Unified Two-Point Fourth Order Family of Iterative Methods

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Abstract

Many variants of existing multipoint methods have been developed. Recently, Khratti et al. (2011) developed a unifying family of two-point fourth order methods which contains the well-known Ostrowski method. The authors also obtained some new methods which are variants of Ostrowski's method. However, it is difficult to compare the methods with the same of the order of convergence. The dynamic behaviour of the methods can be used as a tool for comparison.
In this work, we study the dynamic of six members of the unifying family for some quadratic and cubic polynomials. By means of computer generated plots, we draw their polynomiographs for the polynomials $f(z) = z^2 - 1$ and $f(z) = z^3 - 1$ and explain their respective dynamic behaviour by analyzing the free critical and additional fixed points. Our results show that the methods exhibit different fractal behaviour and the most efficient method based on the size of its basins of attractions was found to the well-known Ostrowski method. This shows that these fourth order variants of Ostrowski's method are inefficient.

Keywords: Fourth order methods, Dynamic behaviour, fractal, Polynomiography, Scaling Theorem, free critical points, additional fixed points, Julia set, Basins of attractions, Efficient method

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1 Introduction

Iterative methods are used to find approximate solutions of nonlinear equations, $f(z) = 0$ which arise from various problems in mathematical and engineering sciences.

The Newton-Raphson method ($2^{nd}NR$) is one of the best known and probably the most used method for solving such nonlinear equations and is given by

$$z_{k+1} = \psi_{2^{nd}NR}(z_k), \tag{1}$$

where

$$\psi_{2^{nd}NR}(z_k) = z_k - u(z_k)$$

and

$$u(z_k) = \frac{f(z_k)}{f'(z_k)}.$$

Several higher order variants of the Newton's method free from second derivatives have been proposed in the literature (see [2] and the references therein). Khratti et al. [7] developed a unifying family of multipoint optimal fourth order methods ($4^{th}UF$) which is given by:

$$z_{k+1} = \psi_{4^{th}UF}(z_k), \tag{2}$$

where

$$\psi_{4^{th}UF}(z_k) = z_k - u(z_k)\left(1 + v(z_k) + 2v(z_k)^2 + \alpha v(z_k)^3 \mathcal{F}(v(z_k))\right),$$

$$v(z_k) = \frac{f(\psi_{2^{nd}NR}(z_k))}{f(z_k)}$$

and

$$\mathcal{F}(v(z_k)) = \sum_{j=0}^{\infty} a_j\ v(z_k)^j,$$

$a_j \in \mathbb{R}$ is a converging power series. This family is termed as two-point methods because its member requires the evaluations of the function at two different points. Khratti et al. [7] proved that the methods are fourth order convergent in the real plane. Their result extends to the complex plane in the following theorem:

Theorem 1. *[7] Let the function $f\colon \mathbf{D} \subset \mathbb{C} \mapsto \mathbb{C}$ has a simple root $z^* \in \mathbf{D}$ in the open interval $\mathbf{D}$. Furthermore the first, second and the third derivatives of the function $f(z)$ belongs in the open interval $\mathbf{D}$. Then the methods of the iterative family (2) are at least fourth order convergent for any choice of the real parameter α and the real power series $\mathcal{F}$. The methods of the family (2) satisfies the error equation*

$$e_{k+1} = -\frac{c_2\left(\left(-5 + \alpha\, a_0\right) {c_2}^2 + c_3 c_1\right) {e_k}^4}{{c_1}^3} + O\left({e_k}^5\right), \tag{3}$$

where the error after k iterations $e_k = z_k - z^*$, the constants $c_k = f^{(j)}(z^*)/j!$ with $j \geq 1$ and a_0 is the coefficient of the power series.

Table 1: Some members of the $4^{th}UF$ family for different selections of α and $\mathcal{F}(v(z))$.

Method	α	$\mathcal{F}(v(z))$	$\psi(z)$
$4^{th}Ost$ [10]	1	$4+18v(z)+16v(z)^2+...$	$z-u(z)\left(\frac{1-v(z)}{1-2v(z)})\right)$
$4^{th}Amit$[9]	1	$1+v(z)+v(z)^2+...$	$z-u(z)\left(v(z)^2+\frac{1}{1-v(z)}\right)$
$4^{th}KLW$ [8]	2	$1+v(z)+v(z)^2+...$	$z-u(z)\left(\frac{1+v(z)^2}{1-v(z)}\right)$
$4^{th}KNS1$ [7]	0	-	$z-u(z)\left(1+v(z)+2v(z)^2\right)$
$4^{th}KNS2$ [7]	5	1	$z-u(z)\left(1+v(z)+2v(z)^2+5v(z)^3\right)$
$4^{th}KNS3$ [7]	1	$\sum_{m=0}^{\infty}\frac{5^{m+1}}{2^m}v(z)^m$	$z-u(z)\left(\frac{2-3v(z)-v(z)^2}{2-5v(z)}\right)$

He also rediscovered some methods including the well-known Ostrowski method ($4^{th}Ost$) and obtained new fourth order methods for different selections of α and $\mathcal{F}(v(z))$. We make a summary of the some of the methods in Table 1. The $4^{th}KNS2$ and $4^{th}KNS3$ have the following error equation

$$e_{k+1}=-\frac{c_2c_3}{{c_1}^2}{e_k}^4+O\left({e_k}^5\right)$$

and are fifth order methods for quadratic functions since $c_3=0$. For this case we term these methods as $5^{th}KNS2q$ and $5^{th}KNS3q$ methods, respectively.
It is difficult to compare these methods because they have same order of convergence. They enjoy their higher order convergence only if the starting point is chosen close to the root. So it is important to find the basins of attractions of the methods by studying their dynamic behaviour in the complex plane.
In this work, we introduce the basic notations and definitions. We prove the Scaling Theorem for the unifying family. We study the dynamic behaviour of its six members for the polynomials, $f(z)=z^2-1$ and $f(z)=z^3-1$ by analyzing their free critical and additional fixed points. Bahman Kalantari [6] coined the term "polynomiography" to be the art and science of visualization in the approximation of roots of polynomial using iteration methods. We draw the polynomiographs of the methods and use them to find the most efficient method based on the size of the basins of attractions.

2 Basic Notations and Preliminaries

Definition 1. *[6, p. 89] A fixed point of the rational function $\mathcal{R}$ is a point z^* such that $\mathcal{R}(z^*) = z^*$.*

Definition 2. *[6, p. 90] Given a fixed point $z^* \in \mathbb{C}$ the quantity $\varphi = \mathcal{R}'(z^*)$ is a well-defined point in $\mathbb{C}$ and is called its multiplier. There are four different basic types of fixed points:*

$$z^* : \begin{cases} \textit{super-attractive, if} \quad |\varphi| = 0 \\ \textit{attractive, if} \quad 0 < |\varphi| < 1 \\ \textit{repelling, if} \quad |\varphi| > 1 \\ \textit{indifferent, if} \quad |\varphi| = 1. \end{cases}$$

An indifferent fixed point is said to be rationally different or parabolic if φ is a root of unity, i.e there exists a natural number $\mathfrak{n}_1$ such that $\varphi^{\mathfrak{n}_1} = 1$, otherwise, irrationally indifferent.

Let z^* be an attracting fixed point of $\mathcal{R}(z)$. Its basin of attraction is the set

$$\mathfrak{B}(z) = \{z \in \widehat{\mathbb{C}} : \mathcal{R}^{\mathfrak{a}}(z) \to z^* \text{ as } \mathfrak{a} \to \infty\}. \tag{4}$$

Definition 3. *[6, p. 123] The Julia set of a given rational function $\mathcal{R}$, denoted by $\mathcal{J}(\mathcal{R})$ is the set of points $z \in \widehat{\mathbb{C}}$ where $\mathcal{R}^{\mathfrak{a}}$ is not normal at the point. The complement $\mathcal{J}(\mathcal{R})$ is called the Fatou set and is denoted by $\mathcal{F}(\mathcal{R})$.*

We list the properties of the Julia set [5]:

1. $\mathcal{J}_{\mathcal{R}}$ is the closure of the repelling periodic points.

2. $\mathcal{J}_{\mathcal{R}}$ is non-empty.

3. $\mathcal{J}_{\mathcal{R}}$ is completely invariant under $\mathcal{R}$; i.e. $\mathcal{R}(\mathcal{J}_{\mathcal{R}}) = \mathcal{J}_{\mathcal{R}} = \mathcal{R}^{-1}(\mathcal{J}_{\mathcal{R}})$.

4. $\mathcal{J}_{\mathcal{R}}$ is the boundary of the basin of attraction of each fixed point or attractive cycle.

5. If $z^* \in \mathcal{J}_{\mathcal{R}}$, then the closure of

$$\{z | \mathcal{R}^{\mathfrak{a}}(z) = z^*, \text{ for some non-negative integer } \mathfrak{a}\},$$

the backward iterates of z^*, is the whole of $\mathcal{J}_{\mathcal{R}}$.

Remark 2. *[5]* **Property 4** *guarantees that, if there are more than two roots, $\mathcal{J}_{\mathcal{R}}$ will be a fractal set.* **Property 1** *guarantees that the Julia set is an unstable set. Iterates of points close to the Julia set will move away from that set. Hence, higher order methods are very sensitive to initial conditions when the initial point is near the Julia set. Nearby points could converge to different roots or might not converge at all. Ideally, if you start with a point actually on the Julia set,* **Property 3** *implies that the iterates will also be on the Julia set. However, in practice, because the Julia set is unstable, the iterates will most likely be thrown off the set because of rounding errors.*

Remark 3. *[4] The zeros which are not zeros of f are referred as additional (or extraneous) fixed points. Their appearance is a striking example of the caution necessary for the selection of starting values of these higher order iteration sequences, which generally lead to smaller basins of attraction. These points are either attracting or repelling but, in any case, they affect the roots' basins of attraction. If the extra fixed points are repelling, they belong to the Julia set, so they cannot trap an iteration sequence.*

To detect the existence of attracting cycles which could interfere with the search for the roots, we observe the orbits of the free critical points of the iteration function.

Definition 4. *[4] Critical values of a function are defined as those values $\mathrm{q} \in \mathbb{C}$ for which $f(z) = \mathrm{q}$ has a multiple root. The multiple root $z = z^*$ is called the critical point of f. This is equivalent to the condition $f'(z^*) = 0$.*

The solutions of the equation $\psi'_{,\mathfrak{q}}(z) = 0$ that are not solutions of the equation of $\mathfrak{q}(z) = 0$ are called free critical points. The free critical points of a higher order method satisfy the equation

$$\frac{\psi'_{IF,\mathfrak{q}}(z)}{\mathfrak{q}(z)^{p-1}} = 0,$$

where p is the order of the method

Theorem 4 (Fatou-Julia)**.** *Let $\psi(z)$ be a rational map. If z_0 is an attracting periodic point, then the immediate basin of attraction $\mathfrak{B}^*(z_0)$ contains at least one critical point.*

Remark 5. *[1] As a consequence of Theorem 4, to detect the existence of an attracting periodic point that interferes with our search of a root of the equation $\mathfrak{q}(z) = 0$, the orbit of each free critical point must be computed and its set of limit point determined. If the set of limit points of the orbit of some free critical points is not a root, that is, a super-attracting fixed point of any iterative method $\psi_{,\mathfrak{q}}(z)$ under consideration, then it must be an attracting periodic orbit.*

Definition 5. *[1] Let f and g be two maps in the Riemann sphere into itself. An analytic conjugacy between f and g is an analytical diffeomorphism $\mathfrak{h}$ from the Riemann sphere onto itself such that $\mathfrak{h} \circ f = g \circ \mathfrak{h}$ (conjugacy equation).*

Conjugacy plays a central role in understanding the behaviour of classes of maps from dynamical system point of view in the sense that it preserves fixed and periodic points and their type as well as basin of attraction.

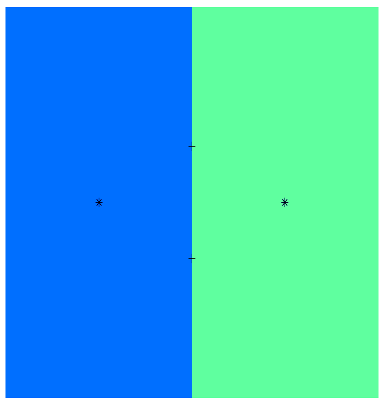
Theorem 6 (Scaling Theorem)**.** *[1] Let $f(z)$ be an analytic function on the Riemann sphere, and let $T(z) = Y_1 z + Y_2$, $Y_1 \neq 0$, be an affine map. If $g(z) = f \circ T(z)$, then $T \circ \psi_{,g} \circ T^{-1} = \psi_{,f}$. That is $\psi_{,f}$ and $\psi_{,g}$ are analytically conjugated by T.*

Remark 7. *[1] If $\widetilde{\mathfrak{q}}(z) = \varsigma_1 \mathfrak{q}(z)$, where ς_1 is a constant, a straightforward calculation shows that $\psi(,\widetilde{\mathfrak{q}}) = \psi(,\mathfrak{q})$, that is, the identity map is a conjugacy between the maps $\psi(,\mathfrak{q})$ and $\psi(,\widetilde{q})$, therefore their dynamics are equivalent. The Scaling Theorem allows us to, modulo suitable changes of coordinates, reduce the study of the dynamics of iterations*

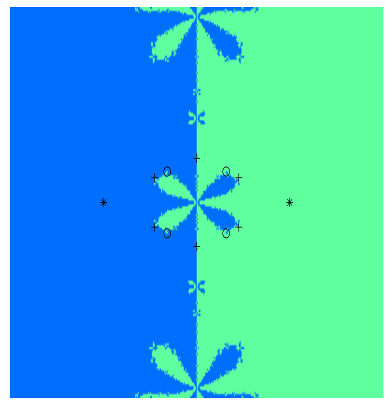
$\psi(,\mathfrak{q})$, *to the study of specific families of iterations of simpler maps. For instance, every quadratic polynomial* $\mathfrak{q}(z) = \mathrm{b}_1 z^2 + \mathrm{b}_2 z + \mathrm{b}_3$ *reduces, via an affine change of coordinates, to a polynomial belonging to the one-parameter family* $p_{\mathrm{c}}(z) = z^2 - \mathrm{c}$, *where* $\mathrm{c} = {\mathrm{b}_2}^2 - 4\mathrm{b}_1\mathrm{b}_3$. *This is nothing but an appropriate re-scaling that puts* $\psi(,\mathfrak{q})$ *inside the conjugacy class of* $\psi(,p_{\mathrm{c}})$ *for some* c.

Theorem 8. *[3, p. 8] Suppose that* $\mathfrak{q}$ *is a polynomial of degree* $\mathrm{d} \geq 2$. *The unit circle* $\mathrm{S}_1 = \{z \in \mathbb{C} : |z| = 1\}$ *is completely invariant if and only if* $\mathfrak{q}(z) = \mathrm{l}z^{\mathrm{d}}$, *where* $|\mathrm{l}| = 1$.

From Theorem 8, $\mathcal{J}(\mathfrak{q}) = \mathrm{S}_1$.
We prove the Scaling Theorem for the 4^{th} *UF* family in the next section.



(a)



(b)

Figure 1: Polynomiographs of 4^{th} *OM* and 4^{th} *Amit* methods for $f(z) = z^2 - 1$

3 Scaling Theorem for the 4^{th} *UF* family

Theorem 9. 4^{th}UF*'s family satisfies the Scaling Theorem, that is,* $T \circ \psi_{4^{th}UF,g} \circ T^{-1}(z) = \psi_{4^{th}UF,f}(z)$.

Proof. It is enough to show that $T \circ \psi_{4^{th}UF,g}(z) = \psi_{4^{th}UF,f} \circ T(z)$.
Since $g(z) = f(T(z))$, we have by induction

$$g^{(\mathrm{a})}(z) = Y_1^{\mathrm{a}} f^{(\mathrm{a})}(T(z)), \text{ for } \mathrm{a} \in \mathbb{N}. \tag{5}$$

We have

$$\begin{aligned} u_g(z) &= \frac{g(z)}{g'(z)} \\ &= \frac{f(T(z))}{Y_1 f'(T(z))}, \text{ using eq. (5) with a} = 1, \\ &= \frac{1}{Y_1} u_f(T(z)). \end{aligned} \tag{6}$$

Then,

$$\begin{aligned} g(\psi_{2^{nd}NR,g})(z) &= g\left(z - \frac{1}{Y_1}u_f(T(z))\right) \\ &= f\left(T\left(z - \frac{1}{Y_1}u_f(T(z))\right)\right). \end{aligned} \tag{7}$$

Now, from the definition of $T(z)$, we have

$$\begin{aligned} T\left(z - \frac{1}{Y_1}u_f(T(z))\right) &= Y_1\left(z - \frac{1}{Y_1}u_f(T(z))\right) + Y_2 \\ &= Y_1 z + Y_2 - u_f(T(z)) \\ &= T(z) - u_f(T(z)). \end{aligned} \tag{8}$$

Using eqs. (7) and (8), we have

$$\begin{aligned} v_g(z) &= \frac{g(\psi_{2^{nd}NR,g}(z))}{g(z)} \\ &= \frac{f\left(T(z) - u_f(T(z))\right)}{f(T(z))} \\ &= v_f(T(z)). \end{aligned} \tag{9}$$

Using eqs. (6) and (9), we finally obtain

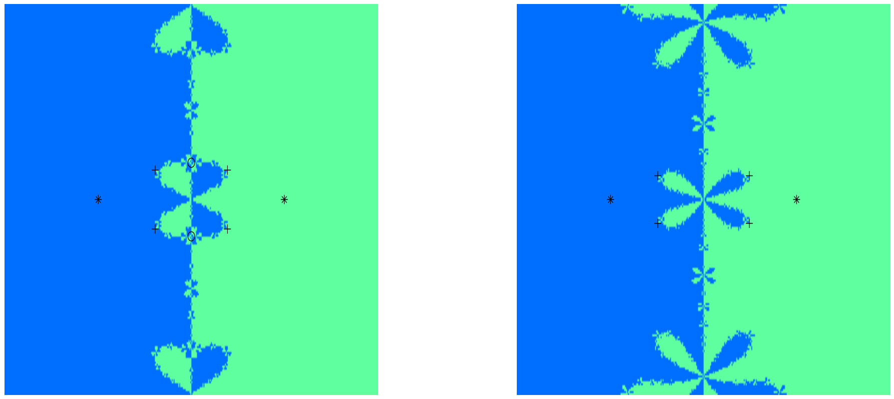
$$\begin{aligned} &T \circ \psi_{4^{th}UF,g}(z) \\ &= Y_1\psi_{4^{th}UF,g}(z) + Y_2 \\ &= Y_1\left[z - u_g(z)\left(1 + v_g(z) + 2v_g(z)^2 + \alpha v_g(z)^3 \sum_{j=0}^{\infty} a_j\ v_g(z)^j\right)\right] + Y_2 \\ &= T(z) - u_f(T(z))\left(1 + v_f(T(z)) + 2v_f(T(z))^2 + \alpha v_f(T(z))^3 \sum_{j=0}^{\infty} a_j\ v_f(T(z))^j\right) \\ &= \psi_{4^{th}UF,f} \circ T(z). \end{aligned}$$

□

4 Study of the members of the 4^{th} *UF* family for the Generic Quadratic Polynomial

In this section, we analyze the dynamic behaviour of the family for the generic quadratic polynomial.

Proposition 10. *For $\mathfrak{q}_c = z^2 - c$, where $c \in \mathbb{C}$, the Julia set of the Ostrowski method, $\mathcal{J}(\psi_{4^{th}Ost,\mathfrak{q}_c})$ is a straight line.*



(a) (b)

Figure 2: Polynomiographs of 4^{th}*KLW* and 4^{th}*KNS1* methods for $f(z) = z^2 - 1$

Proof. Following [4], we apply the Möbius transformation

$$\mathbb{M}(z) = \frac{z + \sqrt{\mathrm{c}}}{z - \sqrt{\mathrm{c}}},$$

the inverse of which is

$$\mathbb{M}^{-1}(z) = \sqrt{\mathrm{c}}\,\frac{z+1}{z-1},$$

so that, using a computer algebra software such as Maple, we have

$$\mathbb{M}\psi_{4^{th}Ost}\mathbb{M}^{-1}(z) = z^4. \tag{10}$$

From Theorem 8, we have $\mathcal{J}(\mathbb{M}\psi_{4^{th}Ost}\mathbb{M}^{-1}(z)) = \mathrm{S}_1$ with interior $\mathfrak{B}(0)$ and exterior $\mathfrak{B}(\infty)$. $\square$

However, since

$$\mathbb{M}\psi_{4^{th}Amit}\mathbb{M}^{-1}(z) = \frac{z^8 + 5z^7 + 10z^6 + 9z^5 + 4z^4}{4z^4 + 9z^3 + 10z^2 + 5z + 1},$$

$$\mathbb{M}\psi_{4^{th}KLW}\mathbb{M}^{-1}(z) = \frac{z^6 + 3z^5 + 3z^4}{3z^2 + 3z + 1},$$

$$\mathbb{M}\psi_{4^{th}KNS1}\mathbb{M}^{-1}(z) = \frac{z^6 + 4z^5 + 5z^4}{5 * z^2 + 4 * z + 1},$$

$$\mathbb{M}\psi_{5^{th}KNS2q}\mathbb{M}^{-1}(z) = \frac{z^8 + 6z^7 + 14z^6 + 14z^5}{14z^3 + 14z^2 + 6z + 1},$$

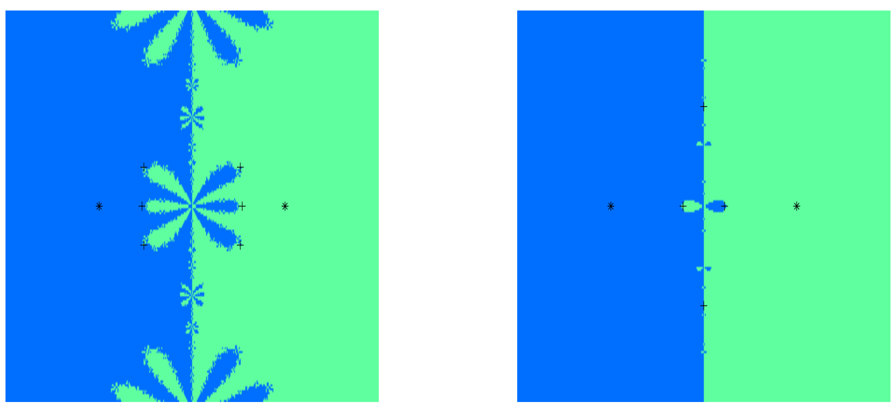
and

$$\mathbb{M}\psi_{5^{th}KNS3q}\mathbb{M}^{-1}(z) = \frac{2z^6 + 3z^5}{3z + 2},$$

the Julia sets of these methods are not straight lines. This implies that only the Ostrowski method will generate an "ideal" fractal. This will be confirmed by a numerical study in the following section.

4.1 Numerical Study for the Quadratic Polynomial $f(z) = \mathfrak{q}_{c=1} = z^2 - 1$.

We now draw the polynomiographs of $f(z) = \mathfrak{q}_{c=1} = z^2 - 1$ with roots $z_1^* = -1$ and $z_2^* = 1$. Let $z_0 = \mathrm{x} + i\ \mathrm{y}$ be the initial point. A square grid of 80000 points, composed of 400 columns and 200 rows corresponding to the pixels of a computer display would represent a region of the complex plane [11]. We consider the square $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$. Each grid point is used as a starting value z_0 of the sequence $z_{k+1} = \psi_{method}(z_k)$ and the number of iterations until convergence is counted for each gridpoint. We assign pale blue colour if the iterates z_k of each grid point converge to the root $z_1^* = -1$ and green colour if they converge to the root $z_2^* = 1$ in at most 100 iterations and if $|z_j^* - z_k| < 0.0001$, $j = 1, 2$. In this way, the basin of attraction $\mathfrak{B}(z_j^*)$ for each root would be assigned a characteristic colour. The common boundaries of these basins of attraction constitute the Julia set of the methods. If the iterates do not satisfy the above criterion for convergence we assign the dark blue colour. The polynomiographs are generated in MATLAB R2010a. Fig. 1 (a) shows the polynomiograph of the $4^{th}OM$ method and we see that its Julia set is the imaginary axis because the $4^{th}OM$ method satisfies Theorem 8. The polynomiographs of the other five methods can be shown in Figs. 1 (a), 2 and 3. It can be observed that their Julia sets are not straight lines because these methods do not satisfy Theorem 8. In figs. 1 to 3, we denote $*$ as the roots, o as the free critical points and $+$ as the additional fixed points.



(a) (b)

Figure 3: Polynomiographs of $5^{th}KNS2q$ and $5^{th}KNS3q$ methods for $f(z) = z^2 - 1$

We denote FCP^o and AFP^+ as the free critical point and additional fixed point of the methods, respectively. We also denote N_o, N_+ and N_D as the number of free critical points, number of additional fixed points and number of diverging starting points, respectively. Table 2 shows a comparison of these numbers. It also gives the values of the free critical and additional fixed points of the methods. All these points are repelling and are found in the Julia set. All starting points converge for the methods considered. The $5^{th}KNS3q$ method is found to be the best of the 5 methods which do not satisfy Theorem

8 as it has no free critical points and 2 additional fixed points which lie on the imaginary axis. Its Julia set is the least complex as shown in 3 (b). For the other methods, the presence of the free critical and additional fixed points may interfere with the root search, thus resulting in complex fractal shapes like petals and hearts as shown in Figs. 1 (b), 2 and 3 (a). The $4^{th}OM$ method is found as the most efficient since it has the largest basins of attractions for the quadratic polynomial.

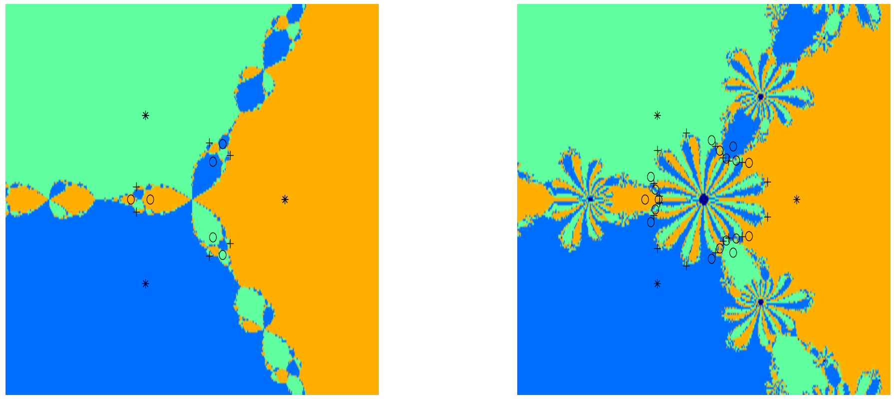
Table 2: Comparison of number of free critical points, additional fixed points and diverging points of the members of UF family for the quadratic polynomial $f(z) = z^2 - 1$.

method	N_o	FCP^o	N_+	AFP^+	N_D
$4^{th}OM$	0	-	2	$\pm 0.5744i$	0
$4^{th}Amit$	4	$\pm 0.3175 \pm 0.3175i$	6	$\pm 0.4535i$, $\pm 0.4535 \pm 0.2542i$	0
$4^{th}KLW$	2	$\pm 0.3780i$	4	$\pm 0.3882 \pm 0.3031i$	0
$4^{th}KNS1$	0	-	4	$\pm 0.4916 \pm 0.2446i$	0
$5^{th}KNS2q$	0	-	6	± 0.5392, $\pm 0.5183 \pm 0.4017i$	0
$5^{th}KNS3q$	0	-	4	$\pm 1.0248i$, ± 0.2239	0

5 Numerical Study for the Cubic Polynomial $f(z) = \mathfrak{q}_{\mathrm{e}=1} = z^3 - 1$.

The dynamic of the $4^{th}UF$ Family for the Generic Cubic Polynomial is rather complex. Therefore we limit ourselves to the numerical study of the Cubic Polynomial $f(z) = \mathfrak{q}_{\mathrm{e}=1} = z^3 - 1$. The roots are $z_1^* = 1$, $z_2^* = -0.5000 + 0.8660i$ and $z_3^* = -0.5000 + 0.8660i$. Each grid point over the region $[-2, 2] \times [-2, 2]$ is coloured accordingly, brownish yellow for convergence to z_1^*, blue for convergence to z_2^* and pale green for convergence to z_3^*. We use the same conditions for convergence as in the quadratic polynomial. Using the conjugate map $\mathcal{S}(z) = \psi\left(\frac{1}{z}\right)$, we can verify that $\psi_{4^{th}UF,\mathfrak{q}_{\mathrm{e}=1}}(\infty) = \infty$ and ∞ is a repelling fixed point of the six members of the $4^{th}UF$ family since $|\mathcal{S}'(0)| > 1$.

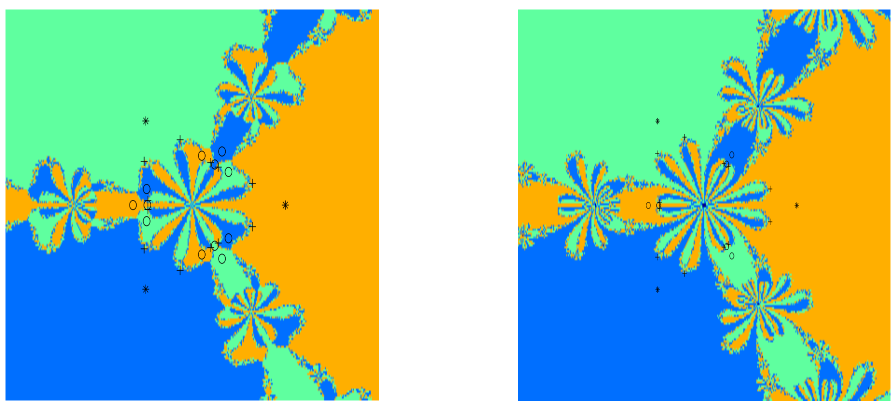
Fig 4 (a) shows the polynomiograph of the $4^{th}OM$ method. There are 6 repelling free critical and 6 repelling additional fixed points. The free critical points are usually on the perpendicular bisector of any two roots. Fig 4 (b) shows the polynomiograph of the $4^{th}Amit$ method. There are 18 free critical and 18 additional fixed points which are all repelling. They are located at the ends of the petals centered at the origin where we can observe some diverging starting points. It is the presence of these repelling points



(a) (b)

Figure 4: Polynomiographs of $4^{th}OM$ and $4^{th}Amit$ methods for $f(z) = z^3 - 1$

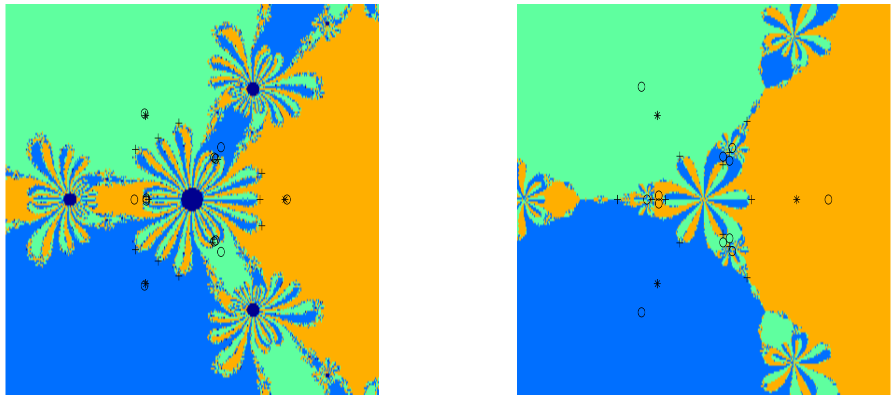
which cause the $4^{th}Amit$ iterates to diverge. Fig 5 (a) shows the polynomiograph of the



(a) (b)

Figure 5: Polynomiographs of $4^{th}KLW$ and $4^{th}KNS1$ methods for $f(z) = z^3 - 1$

$4^{th}KLW$ method and Julia set appears to be butterfly-shaped. There are 12 repelling free critical and 12 repelling additional fixed points for this method. Fig 5 (b) shows the polynomiograph of the $4^{th}KNS1$ method. There are 6 free critical and 12 additional fixed points. These points are repelling and we observe some diverging points at the origin. However, the number of diverging points is less than of the $4^{th}Amit$ method. Fig 6 (a) shows the polynomiograph of the $4^{th}KSN2$ method. There are 12 free critical points and 18 additional fixed points. 3 free critical points (1.0223, $\pm 0.8853i - 0.5111$) are attracting since $\varphi = 0.0016 < 1$ while the rest are repelling. Points near the attracting free critical points converge to the super-attracting fixed points of $f(z)$. All additional fixed points are repelling. It is also observed that this method has the highest number of diverging points because the repelling points surround the origin. Fig 6 (b) shows the polynomiograph of



(a) (b)

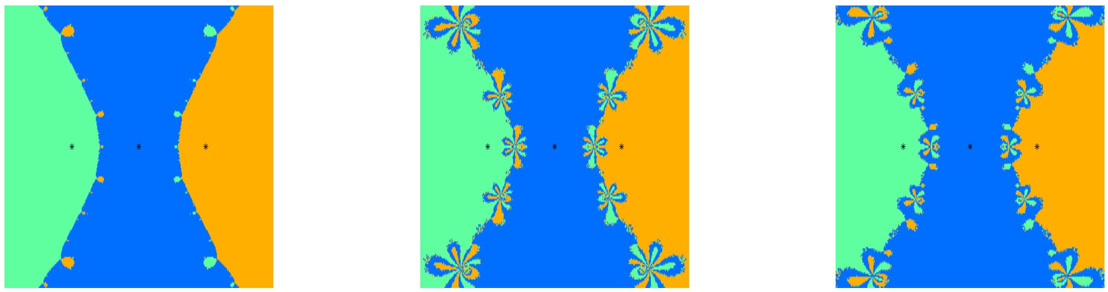
Figure 6: Polynomiographs of 4^{th}*KNS2* and 4^{th}*KNS3* methods for $f(z) = z^3 - 1$

the 4^{th}*KNS3* method. There are 12 free critical and 12 additional fixed points. 3 free critical points (1.3411, $\pm 1.1614i - 0.6705$) are attracting since $\varphi = 0.04 < 1$ while the rest are repelling. All additional fixed points are repelling. However, all starting points are convergent. This is because the repelling free critical points and additional fixed points are mainly located on the perpendicular bisector of any two roots. They do not affect the iterates of nearby points. The 4^{th}*OM* method is again found as the most efficient since it has the largest basins of attractions for this cubic polynomial. Finally, we include

Table 3: Comparison of number of free critical points, additional fixed points and diverging points of the members of *UF* family for the cubic polynomial $f(z) = z^3 - 1$.

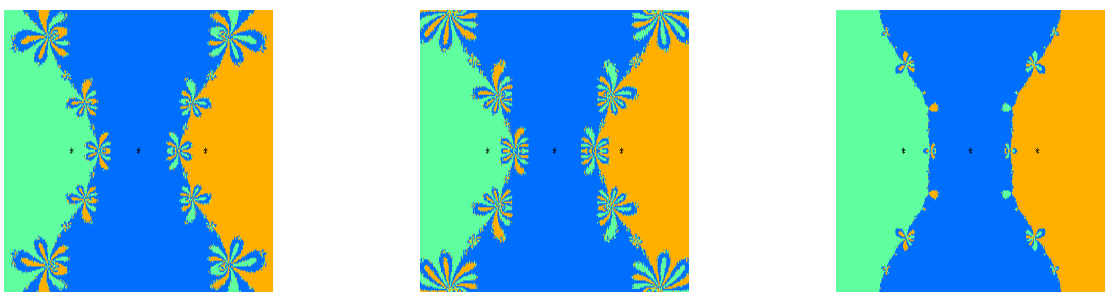
Methods	N_o	N_+	N_D
4^{th}*OM*	6	6	0
4^{th}*Amit*	18	18	110
4^{th}*KLW*	12	12	0
4^{th}*KNS1*	6	12	36
4^{th}*KNS2*	12	18	692
4^{th}*KNS3*	12	12	0

the polynomiographs of the six methods for generic cubic polynomial $f(z) = \mathfrak{q}_{e=0}$ whose roots are $-1,\ 0,\ 1$. They are shown in Figs. 7 and 8. All starting points are convergent for the six methods. We can find bulb and petal shapes in the Julia set. The $4^{th}OM$ method is the most efficient method as it has the smallest Julia set. The $4^{th}Amit$ and $4^{th}KNS2$ methods are the most chaotic methods because the shape of their Julia set are most complex. These observations were also made with the first two polynomials.



(a) (b) (c)

Figure 7: Polynomiographs of $4^{th}OM$, $4^{th}Amit$ and $4^{th}KLW$ methods for $f(z) = z^3 - z$



(a) (b) (c)

Figure 8: Polynomiographs of $4^{th}KNS1$, $4^{th}KNS2$ and $4^{th}KNS3$ methods for $f(z) = z^3 - z$

6 Conclusion

In this work, we prove the Scaling Theorem for the unifying family. We explain the dynamic behaviour of its six members for the polynomials, $f(z) = z^2 - 1$ and $f(z) = z^3 - 1$ by considering their free critical and additional fixed points. We found that these points can interfere with the root search and cause the method to behave chaotically and thus reducing their basins of attractions. We found that the Ostrowski method is the best efficient of the six methods as it behaves the least chaotically and has the largest basins of attractions. We conclude that our analysis on the dynamic behaviour of iterative methods can be used as a tool for comparing methods of same of convergence order using computer generated plots. This enable us to choose the best efficient method from a family.

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