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An Averaging Result for Fuzzy Differential Equations with a Small Parameter

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Abstract: For fuzzy differential equations with a small parameter we prove an averaging result on finite time intervals and under rather weak conditions.

Keywords: Fuzzy differential equations, small parameter, averaging.

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1 Introduction and preliminaries

Fuzzy differential equations have been studied by many authors due to many applications. Reader can refer to the books [12, 16] and the papers [1, 2, 4, 13, 17, 18, 19, 20] and the references therein.

In this work, we prove an averaging result for fuzzy differential equations with a small parameter. As in the previous works of the last author on the justification of the method of averaging for different differential equations (see, for instance, references [7] to [11]), the conditions we assume here are more general than those considered in the literature (compare, for instance, with conditions in [5, 6, 15]).

The structure of the paper is as follows: In Section 2 we present our main result: Theorems 7. We state and prove some preliminary results in Subsection 2.1 and then we give the proof of Theorem 7 in Subsection 2.2. We finish this section with some definitions, notations and properties on fuzzy numbers and maps.

Denote by $\mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ the set of all nonempty compact and convex subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric defined by

$$\rho(A,B) := \max\left(\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right), \quad \forall A,B\in \mathrm{Conv}(\mathbb{R}^{\mathrm{d}}).$$

The metric space $(\mathrm{Conv}(\mathbb{R}^{\mathrm{d}}),\rho)$ is complete.

Let $\mathbb{E}^d$ be the set of all fuzzy numbers, that is, the set of mappings $x:\mathbb{R}^d\to[0,1]$ that satisfy the following conditions:

i) x is normal, that is, there exists $u_0 \in \mathbb{R}^d$ such that $x(u_0) = 1$;

ii) x is fuzzy convex, that is, for any $u, v \in \mathbb{R}^d$ and $\lambda \in [0,1]$, one has

$$x\left(\lambda u + (1-\lambda)v\right) \geq \min\left\{x(u), x(v)\right\};$$

iii) x is upper semicontinuous, that is, for any $u_0 \in \mathbb{R}^d$ and $\varepsilon > 0$, there exists $\delta = \delta(u_0, \varepsilon) > 0$ such that $x(u) < x(u_0) + \varepsilon$ for all $u \in \mathbb{R}^d$ that satisfy the condition $|u - u_0| < \delta$;

iv) the closure of the set $\{u \in \mathbb{R}^d : x(u) > 0\}$ is compact.

The zero element in $\mathbb{E}^d$ is defined by $\hat{0}(u) = 1$ for $u = 0$ and 0 otherwise.

For $\alpha \in (0,1]$, the α-section $[x]^\alpha$ of a mapping $x \in \mathbb{E}^d$ is defined as the set $\{u \in \mathbb{R}^d : x(u) \geq \alpha\}$. The zero section of a mapping $x \in \mathbb{E}^d$ is defined as the closure of the set $\{u \in \mathbb{R}^d : x(u) > 0\}$.

For any $\alpha \in [0,1]$, $[x]^\alpha \in \mathrm{Conv}(\mathrm{R}^\mathrm{d})$.

The addition and scalar multiplication for the fuzzy numbers are defined as follows:

$$[x+y]^\alpha = [x]^\alpha + [y]^\alpha,\ [\lambda x]^\alpha = \lambda [x]^\alpha,\ x, y \in \mathbb{E}^d,\ \lambda \in \mathbb{R},\ \alpha \in [0,1].$$

The set of fuzzy numbers is a convex cone under the addition and scalar multiplication.

The metric in $\mathbb{E}^d$ is defined by

$$D(x,y) = \sup_{\alpha \in [0,1]} \rho([x]^\alpha, [y]^\alpha),\ \forall x, y \in \mathbb{E}^d$$

where ρ is the Hausdorff metric and D is such that:

i) $(\mathbb{E}^d, D)$ is a complete metric space;

ii) $D(x+z, y+z) = D(x,y)$ for all $x, y, z \in \mathbb{E}^d$;

iii) $D(kx, ky) = |k| D(x,y)$ for all $x, y \in \mathbb{E}^d$ and $k \in \mathbb{R}$.

The following definitions and propositions are given in [3, 14]. Let I be an interval in $\mathbb{R}$.

Definition 1 *Let $h : I \to \mathbb{E}^d$. The integral of h over I, denoted by $\int_I h(t)dt$, is defined by*

$$\begin{aligned}\left[\int_I h(t)dt\right]^\alpha &= \int_I h_\alpha(t)dt \\ &= \left\{\int_I \phi(t)dt \mid \phi : I \to \mathbb{R}^d \text{ is a measurable selection for } h_\alpha\right\},\end{aligned}$$

for all $\alpha \in (0,1]$.

A strongly measurable and integrably bounded mapping $h : I \to \mathbb{E}^d$ is said to be integrable over I if $\int_I h(t)dt \in \mathbb{E}^d$.

Proposition 2 *Let $h, g : I \to \mathbb{E}^d$ be integrable and $\lambda \in \mathbb{R}$. Then*

i) $\displaystyle\int_I \big(h(t) + g(t)\big)dt = \int_I h(t)dt + \int_I g(t)dt;$

ii) $\displaystyle\int_I \lambda h(t)dt = \lambda \int_I h(t)dt;$

iii) $\displaystyle D\left(\int_I h(t)dt, \int_I g(t)dt\right) \le \int_I D(h(t), g(t))dt.$

Definition 3 *A mapping $h : I \to \mathbb{E}^d$ is continuous at $t_0 \in I$ if, for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $D(h(t), h(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta$, $t \in I$.*

A mapping $h : I \to \mathbb{E}^d$ is continuous on I if it is continuous at every $t_0 \in I$.

Proposition 4 *If $h : I \to \mathbb{E}^d$ is continuous, then it is integrable.*

Definition 5 *A mapping $h : I \times \mathbb{E}^d \to \mathbb{E}^d$ is continuous at $(t_0, x_0) \in I \times \mathbb{E}^d$ if, for any $\varepsilon > 0$ there exists $\delta = \delta(t_0, x_0, \varepsilon) > 0$ such that $D(h(t, x), h(t_0, x_0)) < \varepsilon$ whenever $|t - t_0| < \delta$ and $D(x, x_0) < \delta$, $t \in I$ and $x \in \mathbb{E}^d$.*

A mapping $h : I \times \mathbb{E}^d \to \mathbb{E}^d$ is continuous on $I \times \mathbb{E}^d$ if it is continuous at every $(t_0, x_0) \in I \times \mathbb{E}^d$.

Let $x, y \in \mathbb{E}^d$. If there exists a $z \in \mathbb{E}^d$ such that $x = y + z$, then we call z the H-difference of x and y, denoted by $x - y$.

Definition 6 *A mapping $h : I \to \mathbb{E}^d$ is differentiable at $t_0 \in I$ if there exists a $\dot{h}(t_0) \in \mathbb{E}^d$ such that the limits*

$$\lim_{\Delta \to 0^+} \frac{h(t_0 + \Delta) - h(t_0)}{\Delta} \qquad \text{and} \qquad \lim_{\Delta \to 0^+} \frac{h(t_0) - h(t_0 - \Delta)}{\Delta}$$

exist and equal to $\dot{h}(t_0)$.

A mapping $h : I \to \mathbb{E}^d$ is called differentiable on I if it is differentiable at every $t_0 \in I$.

Here the limit is taken in the metric space $(\mathbb{E}^d, D)$. At the end points of I, we consider only the one-sided derivatives.

If $h : I \to \mathbb{E}^d$ is differentiable at $t_0 \in I$, then we say that $\dot{h}(t_0)$ is the fuzzy derivative of h at t_0.

2 Main result

Consider the following initial value problem associated to a fuzzy differential equation with a small parameter

$$x' = f\left(\frac{t}{\varepsilon}, x\right), \; x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$, $\mathbb{U}$ is an open subset of $\mathbb{E}^d$, $x_0 \in \mathbb{U}$ and $\varepsilon > 0$ is a small parameter.

We associate (1) with the averaged initial value problem

$$y' = f^o(y), \;\; y(0) = x_0, \tag{2}$$

where the mapping $f^o : \mathbb{U} \to \mathbb{E}^d$ is such that, for any $x \in \mathbb{U}$

$$\lim_{T\to\infty} D\left(\frac{1}{T}\int_0^T f(\tau, x)d\tau, f^o(x)\right) = 0. \tag{3}$$

Theorem 7 *Suppose that the following hold:*

(H1) the mapping $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$ in (1) is continuous;

(H2) there exist a locally Lebesgue integrable mapping $m : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $M > 0$ such that

$$D(f(t, x), \hat{0}) \leq m(t), \quad \forall t \in \mathbb{R}_+, \forall x \in \mathbb{U}$$

with

$$\int_{t_1}^{t_2} m(t)dt \leq M(t_2 - t_1), \;\; \forall t_1, t_2 \in \mathbb{R}_+;$$

(H3) there exists a constant $\lambda > 0$ such that for all continuous mappings $u, v : \mathbb{R}_+ \to \mathbb{U}$ and all $t_1, t_2 \in \mathbb{R}_+$, $t_1 \leq t_2$,

$$D\left(\int_{t_1}^{t_2} f(\tau, u(\tau))d\tau, \int_{t_1}^{t_2} f(\tau, v(\tau))d\tau\right) \leq \lambda \int_{t_1}^{t_2} D(u(\tau), v(\tau))d\tau; \tag{4}$$

(H4) for all $x \in \mathbb{U}$, the limit (3) exists.

Let $x_0 \in \mathbb{U}$. Let x_ε be a solution of (1) and $I_\varepsilon = [0, \omega_\varepsilon)$, $0 < \omega_\varepsilon \leq \infty$, its maximal positive interval of definition. Let y be the (unique) solution of (2) and $J = [0, \omega_0)$, $0 < \omega_0 \leq \infty$, its maximal positive interval of definition. Then, for any $L > 0$, $L \in I_\varepsilon \cap J$, and $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(x_0, L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the following condition is satisfied:

$$D(x_\varepsilon(t), y(t)) < \delta, \quad \forall t \in [0, L].$$

Remark 8 *Using condition (H3) we will prove in Lemma 9 below that the mapping $f^o : \mathbb{U} \to \mathbb{E}^d$ in (3) is Lipschitz continuous so that the uniqueness of the solution of the averaged initial value problem (2) is guaranteed.*

Notice that condition (H3) is a Lipschitz-type condition on the indefinite integral of f and not on f itself. On the other hand, the averaging results stated in [5, 6, 15] are proved under conditions that are stronger compared to the ones above. In particular, the authors assume that the mapping f is uniformly bounded and is Lipschitz continuous to respect to the second variable.

2.1 Technical lemmas

Here we prove some results we need for the proof of Theorem 7.

Lemma 9 *Let $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$. Suppose that the mapping f satisfies conditions (H2)-(H4) in Theorem 7. Then the mapping $f^0 : \mathbb{U} \to \mathbb{R}^d$ in (3) is uniformly bounded by constant M in condition (H2), that is, $D(f^o(x), \hat{0}) \leq M$, for all $x \in \mathbb{U}$, and satisfies the Lipschitz condition with constant λ as in condition (H3).*

Proof. Boundedness of f^o by M. Let $x \in \mathbb{U}$. By conditions (H2) and (H4) we deduce that, for any $\eta > 0$ there exists $T_0 = T_0(x, \eta) > 0$ such that, for all $T \geq T_0$ we have

$$\begin{aligned} D(f^o(x), \hat{0}) &\leq D\left(f^o(x), \frac{1}{T}\int_0^T f(\tau, x)d\tau\right) + D\left(\frac{1}{T}\int_0^T f(\tau, x)d\tau, \hat{0}\right) \\ &\leq \eta + \frac{1}{T}\int_0^T D\left(f(\tau, x), \hat{0}\right) d\tau \leq \eta + M. \end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain the desired result.

Lipschitz condition of f^o. Let $x, x' \in \mathbb{U}$. By conditions (H3) and (H4) we can easily deduce that, for any $\eta > 0$ there exists $T_0 = T_0(x, x', \eta) > 0$ such that, for all $T \geq T_0$ we have

$$\begin{aligned} D(f^o(x), f^o(x')) \leq D\left(f^o(x), \frac{1}{T}\int_0^T f(\tau, x)d\tau\right) \\ + \frac{1}{T} D\left(\int_0^T f(\tau, x)d\tau, \int_0^T f(\tau, x')d\tau\right) \\ + D\left(f^o(x'), \frac{1}{T}\int_0^T f(\tau, x')d\tau\right) \\ \leq 2\eta + \frac{1}{T}\lambda \int_0^T D(x, x')d\tau = 2\eta + \lambda D(x, x'). \end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain that

$$D(f^o(x), f^o(x')) \leq \lambda D(x, x').$$

This finishes the proof. □

Lemma 10 *Let $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$. Suppose that the mapping f satisfies conditions (H1), (H2) and (H4) in Theorem 7. Then, for all $x \in \mathbb{U}$, $t \geq 0$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0^+} D\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Proof. Let $x \in \mathbb{U}$, $t \geq 0$ and $\alpha > 0$.
Case 1: $t = 0$. From condition (H4), it follows immediately that

$$\lim_{\varepsilon \to 0} D\left(\frac{\varepsilon}{\alpha}\int_0^{\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Case 2: $t \in (0, L]$. It is not difficult to verify that

$$\begin{aligned} D&\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) \\ &\leq D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) \\ &\quad + \frac{L}{\alpha}\left[D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right)\right. \\ &\quad \left. + D\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau, f^o(x)\right)\right]. \end{aligned} \tag{5}$$

From condition (H4), we can easily deduce that

$$\lim_{\varepsilon \to 0} D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0$$

and

$$\lim_{\varepsilon \to 0} D\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Therefore the right-hand side of (5) tends to zero as $\varepsilon \to 0^+$ and the result is proved. □

The next corollary follows directly from Lemma 10.

Corollary 11 *Suppose that the mapping f in (1) satisfies conditions (H1)-(H4) in Theorem 7. Let $x_0 \in \mathbb{U}$. Let y be the (unique) solution of (2) and $J = [0, \omega_0)$, $0 < \omega_0 \leq \infty$, its maximal positive interval of definition. Let $L > 0$ such that $L \in J$. Then, for all $t \in [0, L]$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0} D\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, y(t))d\tau, f^o(y(t))\right) = 0. \tag{6}$$

Lemma 12 *Suppose that the mapping f in (1) satisfies conditions (H1)-(H4) in Theorem 7. Let $x_0 \in \mathbb{U}$. Let y be the (unique) solution of (2) and $J = [0, \omega_0)$, $0 < \omega_0 \leq \infty$, its maximal positive interval of definition. Then, for all $L > 0$ such that $L \in J$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,L]} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right)d\tau, \int_0^t f^o(y(\tau))d\tau\right) = 0.$$

Proof. Let $L > 0$, $L \in J$, and $t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_p = L$, $p \in \mathbb{N}$, a partition of $[0, L]$ with $\alpha = \alpha(\varepsilon) := t_{n+1} - t_n$, $n = 1, \ldots, p$ and $\lim_{\varepsilon \to 0} \alpha = 0$. Let $t \in [t_m, t_{m+1}]$ for any $m \in \{0, \cdots, p-1\}$. Then

$$\begin{aligned} D&\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_0^t f^o(y(\tau))d\tau\right) \\ &\leq \sum_{n=0}^{m-1} D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(y(\tau))d\tau\right) \\ &\quad + D\left(\int_{t_m}^{t} f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_{t_m}^{t} f^o(y(\tau))d\tau\right). \end{aligned} \tag{7}$$

By condition (H2) and Lemma 9 we have

$$\begin{aligned} D&\left(\int_{t_m}^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_{t_m}^t f^o(y(\tau))d\tau\right) \\ &\leq D\left(\int_{t_m}^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \hat{0}\right) + D\left(\int_{t_m}^t f^o(y(\tau))d\tau, \hat{0}\right) \\ &\leq \int_{t_m}^t D\left(f\left(\frac{\tau}{\varepsilon}, y(\tau)\right), \hat{0}\right) d\tau + \int_{t_m}^t D\left(f^o(y(\tau)), \hat{0}\right) d\tau \leq 2M\alpha. \end{aligned}$$

Now, for each $n = 0, \ldots, m-1$ and $\tau \in [t_n, t_{n+1}]$, by Lemma 10 (boundedness of f^o par constant M) we can easily deduce that $D(y(\tau), y(t_n)) \leq M\alpha$ so that by condition (H3) and Lemma 10 (Lipschitz condition of f^o), it follows, respectively, that

$$\begin{aligned} D&\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, y(t_n)\right) d\tau\right) \\ &\leq \lambda \int_{t_n}^{t_{n+1}} D(y(\tau), y(t_n))d\tau \leq \lambda M\alpha^2 \end{aligned}$$

and

$$\begin{aligned} D&\left(\int_{t_n}^{t_{n+1}} f^o(y(\tau))d\tau, \int_{t_n}^{t_{n+1}} f^o(y(t_n))d\tau\right) \\ &\leq \int_{t_n}^{t_{n+1}} D\left(f^o(y(\tau)), f^o(y(t_n))\right) d\tau \\ &\leq \lambda \int_{t_n}^{t_{n+1}} D(y(\tau), y(t_n))d\tau \leq \lambda M\alpha^2. \end{aligned}$$

Hence, from (7), it follows that

$$\begin{aligned} D&\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_0^t f^o(y(\tau))d\tau\right) \\ &\leq \sum_{n=0}^{m-1} D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, y(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(y(t_n))d\tau\right) \\ &\quad + \sum_{n=0}^{m-1} 2\lambda M\alpha^2 + 2M\alpha. \end{aligned} \tag{8}$$

For each $n = 0, \dots, m-1$, we have

$$\begin{aligned}\beta_n &:= D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, y(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(y(t_n))d\tau\right)\\ &= \alpha D\left(\frac{\varepsilon}{\alpha}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha/\varepsilon} f\left(\tau, y(t_n)\right) d\tau, f^o(y(t_n))\right) := \alpha\varrho_n \le \alpha\varrho_m\end{aligned}$$

where $\varrho_m = \max\{\varrho_n = \varrho_n(\varepsilon) : n = 0, \dots, m-1\}$ and, by Corollary 11, $\lim\limits_{\varepsilon\to 0} \varrho_n = 0$.

Then

$$\sum_{n=0}^{m-1} \beta_n \le \varrho_m \sum_{n=0}^{m-1} \alpha = \varrho_m \sum_{n=0}^{m-1} (t_{n+1} - t_n) = \varrho_m t \le \varrho_m L \le \varrho L,$$

where $\varrho = \varrho(\varepsilon) = \max\{\varrho_m : m = 0, \dots, p-1\}$ and $\lim\limits_{\varepsilon\to 0} \varrho = 0$.

On the other hand, we have

$$\sum_{n=0}^{m-1} 2\lambda M\alpha^2 = 2\lambda M\alpha \sum_{n=0}^{m-1} \alpha \le 2\lambda M\alpha t \le 2\lambda M\alpha L.$$

Finally, from (8) we obtain

$$\sup_{t\in[0,L]} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_0^t f^o(y(\tau))d\tau\right) \le 2M(\lambda L + 1)\alpha. \tag{9}$$

As the right-hand side of (9) tends to zero as $\varepsilon \to 0^+$, the lemma is proved. □

2.2 Proof of Theorem 7

We assume that the conditions in Theorem 7 hold.

For $t \in [0, L] \subset I_\varepsilon \cap J$, using condition (H3), we obtain

$$\begin{aligned}D(y(t), x_\varepsilon(t)) &= D\left(\int_0^t f^o(y(\tau))d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau\right)\\ &\le D\left(\int_0^t f^o(y(\tau))d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau\right)\\ &\quad + D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau\right)\\ &\le \sigma + \lambda\int_0^t D\left(y(\tau), x_\varepsilon(\tau)\right) d\tau\end{aligned} \tag{10}$$

where

$$\sigma = \sigma(\varepsilon) := \sup_{t\in[0,L]} D\left(\int_0^t f^o(y(\tau))d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau\right).$$

By Lemma 12, we have $\lim\limits_{\varepsilon\to 0} \sigma = 0$.

By Gronwall-Bellman's Lemma, from (10) it follows that

$$D(y(t), x_\varepsilon(t)) \leq \sigma e^{\lambda t} \leq \sigma e^{\lambda L}$$

from which we deduce that

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,L]} D(x_\varepsilon(t), y(t)) = 0.$$

The proof is complete. □

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