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Convergence of Nonlinear Recurrence Relations with Threshold Control and 3-Periodic Coefficients

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Abstract: We study the difference equation $x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1})$ as a model for a single neuron, where $a_i \in (0,1), b_i = 1 - a_i, i = 0,1,2$ and f satisfies the McCulloch-Pitts nonlinearity. It is found that each solution tends to -1 or 1, depending on whether the parameter λ varies from $-\infty$ to $+\infty$. We hope that our results will be useful in understanding interacting network models involving piecewise constant control functions.

Keywords: Recurrent equations, Periodic Coefficients, Convergence

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1 Introduction

Let $\mathbb{N} = \{0,1,2,\cdots\}$. In [1], Chen considers the equation

$$x_n = x_{n-1} + g(x_{n-k}), n \in \mathbb{N},$$

where k is a nonnegative integer and $g : \mathbb{R} \to \mathbb{R}$ is a McCulloch-Pitts type function

$$g(\xi) = \begin{cases} -1 & u \in (\sigma, +\infty) \\ 1 & u \in (-\infty, \sigma] \end{cases},$$

in which σ is a constant which acts as a threshold. It is shown that every solution is truncated periodic with the minimal period $2(2l+1)$ for some $l \geq 0$ such that $(k-l)/(2l+1)$ is a nonnegative even integer. Yet in real life models, the coefficients a and b, since they are a part of the control mechanism, can rarely be kept constants. They may became time dependent and show periodic behaviors. For this reason, in [2], the authors discussed the limit cycles of the following difference equation

$$x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1}), n \in \mathbb{N}, \tag{1}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are 2-periodic sequences with $a_i \in (0,1), b_i \in (0,+\infty), i = 0, 1$. And $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear threshold function of the form

$$f_\lambda(u) = \begin{cases} 1 & x \in (0, \lambda] \\ 0 & x \in (-\infty, 0) \bigcup (\lambda, -\infty) \end{cases},$$

by the transform $x_{2n} = y_n, x_{2n+1} = z_n$ for $n \in \{-1, 0, \cdots\}$, the above equation can be converted into the following 2−dimensional autonmous dynamical system

$$\begin{cases} y_n = a_0 y_{n-1} + b_0 f_\lambda(z_{n-1}) \\ z_n = a_1 z_{n-1} + b_1 f_\lambda(y_n) \end{cases}, \tag{2}$$

in which the positive number λ can be regarded as a threshold bifurcation parameter. By induction, all solutions of (2) from $(-\infty, 0]^2$ tend to the point (0,0), all solutions of (2) from $\mathbb{R}^2/(-\infty, 0]^2$ tend to the point $(\frac{b_0}{1-a_0}, 0), (0, \frac{b_1}{1-a_1})$, or $(\frac{b_0}{1-a_0}, \frac{b_1}{1-a_1})$.

This paper mainly studies the following form of nonlinear difference equation

$$u_n = a_n u_{n-2} + b_n f_\lambda(u_{n-1}), n \in \mathbb{N}, \tag{3}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are 3-periodic sequences with $a_i \in (0,1)$, $b_i = 1 - a_i$, $i = 0, 1, 2$. f satisfies

$$f_\lambda(u) = \begin{cases} 1 & u \in (\lambda, +\infty) \\ -1 & u \in (-\infty, \lambda] \end{cases},$$

in which the number λ can be regarded as a threshold bifurcation parameter. Note that given $(x_{-2}, x_{-1}) \in \mathbb{R}^2$, we may compute from (3) the numbers $x_0, x_1, x_2, \cdots$ in a unique manner. The corresponding sequence $\{x_n\}_{n=-2}^{\infty}$ is called the solution of (3), determined by or originated from the initial vector (x_{-2}, x_{-1}).

Recall also that a positive integer η is a period of the sequence $\{\omega_n\}_{n=\alpha}^{\infty}$ if $\omega_{\eta+n} = \omega_n$ for all $n \geq \alpha$ and that τ is the least or prime period of $\{\omega_n\}_{n=\alpha}^{\infty}$ if τ is the least among all periods of $\{\omega_n\}_{n=\alpha}^{\infty}$. The sequence $\{\omega_n\}_{n=\alpha}^{\infty}$ is said to be τ-periodic if τ is the least period. The sequence $\omega = \{\omega_n\}_{n=\alpha}^{\infty}$ is said to be asymptotically periodic if there exist real numbers $\omega^{(0)}, \omega^{(1)}, \cdots, \omega^{(\omega-1)}$, where ω is a positive integer, such that $\lim_{n\to\infty} \omega_{\omega n+i} = \omega^{(i)}, i = 0, 1, \cdots, \omega - 1$. In case

$$\{\omega^{(0)}, \omega^{(1)}, \cdots, \omega^{(\omega-1)}, \omega^{(0)}, \omega^{(1)}, \cdots, \omega^{(\omega-1)}, \cdots\}$$

is an ω-periodic sequence, we say that ω is an asymptotically ω-periodic sequence tending to the limit ω-cycle. In particular, an asymptotically 1-periodic sequence is a convergent sequence and conversely.

By the transform $u_{3n} = x_n, u_{3n+1} = y_n, u_{3n+2} = z_n$ the above equation (3) can be converted into the following 3−dimensional autonmous dynamical system

$$\begin{cases} x_n = a_0 y_{n-1} + b_0 f_\lambda(z_{n-1}) \\ y_n = a_1 z_{n-1} + b_1 f_\lambda(x_n) \\ z_n = a_2 x_n + b_2 f_\lambda(y_n) \end{cases} . \tag{4}$$

Therefore, all the asymptotic properties of (3) can be obtained from those of (4). Before doing so, let us recall that if given a real initial vector (y_{-1}, z_{-1}), we may generate a real vector sequence $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ which is called a solution of (4). Therefore, to obtain complete asymptotic behaviors of (4), we need to derive the results for solutions of (4) determined by vectors in the entire plane. In the following discussion, we will allow the bifurcation parameter λ to vary from $-\infty$ to $+\infty$. Indeed, we will consider five cases: (i) $\lambda = 1$, (ii) $\lambda > 1$, (iii) $\lambda < -1$, (iv) $\lambda = -1$, and (v) $-1 < \lambda < 1$.
For the sake of convenience, we also need to introduce some notations: $\sigma, A_i^{\pm}, B_{i,j}^{\pm}, C_{i,k}^{\pm}$.

$$\sigma = a_0 a_1 a_2, A_i^{\pm} = \frac{\lambda \pm 1 \mp \sigma^i}{\sigma^i}, B_{i,j}^{\pm} = \frac{\lambda \pm 1 \mp a_j \sigma^i}{a_j \sigma^i}, C_{i,k}^{\pm} = \frac{\lambda \pm 1 \mp a_0 a_k \sigma^i}{a_0 a_k \sigma^i} \tag{5}$$

where $i \in \mathbb{N},\ j = 0, 1.\ k = 1, 2$.
When $-1 \le \lambda \le 1$, we can give the partition:
$\lambda = A_0^+ < A_1^+ < B_{1,j}^+ < C_{1,k}^+ < \cdots < A_i^+ < B_{i,j}^+ < C_{i,k}^+ < \cdots < A_{i+1}^+ < \cdots$. And $\lim_{i\longrightarrow\infty} A_{i+1}^+ = +\infty$.
Let

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty} \{(A_i^+, B_{i,0}^+] \bigcup (B_{i,0}^+, C_{i,2}^+] \bigcup (C_{i,2}^+, A_{i+1}^+]\}, \tag{6}$$

and

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty} \{(A_i^+, B_{i,1}^+] \bigcup (B_{i,1}^+, C_{i,1}^+] \bigcup (C_{i,1}^+, A_{i+1}^+]\}. \tag{7}$$

When $\lambda > 1$, we can give the partition:
$\lambda = A_0^- < A_1^- < B_{1,j}^- < C_{1,k}^- < \cdots < A_i^- < B_{i,j}^- < C_{i,k}^- < \cdots < A_{i+1}^- < \cdots$. And $\lim_{i\longrightarrow\infty} A_{i+1}^- = +\infty$.
Let

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty}\{(A_i^-, B_{i,0}^-]\bigcup(B_{i,0}^-, C_{i,2}^-]\bigcup(C_{i,2}^-, A_{i+1}^-]\}, \tag{8}$$

and

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty}\{(A_i^-, B_{i,1}^-]\bigcup(B_{i,1}^-, C_{i,1}^-]\bigcup(C_{i,1}^-, A_{i+1}^-]\}. \tag{9}$$

When $\lambda < 1$, we can give the partition:
$A_{i+1}^- < C_{i,k}^- < B_{i,j}^- < A_i^- < C_{i-1,k}^- < \cdots < A_0^- = \lambda$. And $\lim_{i\longrightarrow\infty} A_{i+1}^- = -\infty$.
Let

$$(-\infty, \lambda] = \bigcup_{i=0}^{\infty}\{(A_{i+1}^-, C_{i,2}^-]\bigcup(C_{i,2}^-, B_{i,0}^-]\bigcup(B_{i,0}^-, A_i^-]\}, \tag{10}$$

and

$$(-\infty, \lambda] = \bigcup_{i=0}^{\infty}\{(A_{i+1}^-, C_{i,1}^-]\bigcup(C_{i,1}^-, B_{i,1}^-]\bigcup(B_{i,1}^-, A_i^-]\}. \tag{11}$$

When $\lambda < -1$, we can give the partition:
$A_{i+1}^+ < C_{i,k}^+ < B_{i,j}^+ < A_i^+ < C_{i-1,k}^+ < \cdots < A_0^+ = \lambda$. And $\lim A_{i+1}^+ = -\infty$.
Let

$$(-\infty, \lambda] = \bigcup_{i=0}^{\infty}\{(A_{i+1}^+, C_{i,2}^+]\bigcup(C_{i,2}^+, B_{i,0}^-]\bigcup(B_{i,0}^+, A_i^+]\}, \tag{12}$$

let

$$(-\infty, \lambda] = \bigcup_{i=0}^{\infty}\{(A_{i+1}^+, C_{i,1}^+]\bigcup(C_{i,1}^+, B_{i,1}^+]\bigcup(B_{i,1}^+, A_i^+]\}. \tag{13}$$

2 Main Results

The case where $\lambda = 1$.

Lemma 1. Suppose $\lambda = 1$. If $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is a solution of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$, then there exists an integer $m \in \{-1, 0, \cdots\}$ such that $(y_m, z_m) \in (-\infty, \lambda]^2$.

Proof. (i). Suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$, then we are done. (ii). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda]$. By (4), (5), (6) and induction, its limiting behavior can be summarized in the following table:

Table 1:

$y_{-1} \in (A_i^+, B_{i,0}^+]$	$z_{-1} \in (-\infty, \lambda]$	$(y_{2i}, z_{2i}) \in (-\infty, \lambda]^2$
$y_{-1} \in (B_{i,0}^+, C_{i,2}^+]$	$z_{-1} \in (-\infty, \lambda]$	$(y_{2i}, z_{2i}) \in (-\infty, \lambda]^2$
$y_{-1} \in (C_{i,2}^+, A_{i+1}^+]$	$z_{-1} \in (-\infty, \lambda]$	$(y_{2i+1}, z_{2i+1}) \in (-\infty, \lambda]^2$

For instance, the first row states if $(y_{-1}, z_{-1}) \in (A_i^+, B_{i,0}^+] \times (-\infty, \lambda]$.

$$\begin{aligned}
x_0 &= a_0 y_{-1} + b_0 f_\lambda(z_{-1}) = a_0 y_{-1} - b_0 \in (\frac{\lambda + 1 - a_1 a_2 \sigma^{i-1}}{a_1 a_2 \sigma^{i-1}}, \frac{\lambda + 1 - \sigma^i}{\sigma^i}], \\
y_0 &= a_1 z_{-1} + b_1 f_\lambda(x_0) = a_1 z_{-1} + b_1 \le a_1 \lambda + b_1 = \lambda, \\
z_0 &= a_2 x_0 + b_2 f_\lambda(y_0) = a_2 x_0 - b_2 \in (\frac{\lambda + 1 - a_1 \sigma^{i-1}}{a_1 \sigma^{i-1}}, \frac{\lambda + 1 - a_0 a_1 \sigma^{i-1}}{a_0 a_1 \sigma^{i-1}}], \\
x_1 &= a_0 y_0 + b_0 f_\lambda(z_0) = a_0 y_0 + b_0 \le a_0 \lambda + b_0 = \lambda, \\
y_1 &= a_1 z_0 + b_1 f_\lambda(x_1) = a_1 z_0 - b_1 \in (A_{i-1}^+, B_{i-1,0}^+], \\
z_1 &= a_2 x_1 + b_2 f_\lambda(y_1) = a_2 x_1 + b_2 \le a_2 \lambda + b_2 = \lambda, \\
&\vdots \\
y_{2i-1} &= a_1 z_{2i-2} + b_1 f_\lambda(x_{2i-1}) = a_1 z_{2i-2} - b_1 \in (A_0^+, B_{0,0}^+], \\
z_{2i-1} &= a_2 x_{2i-1} + b_2 f_\lambda(y_{2i-1}) = a_2 x_{2i-1} + b_2 \in (-\infty, \lambda], \\
x_{2i} &= a_0 y_{2i-1} + b_0 f_\lambda(z_{2i-1}) = a_0 y_{2i} - b_0 \in (a_0 \lambda - b_0, \lambda], \\
y_{2i} &= a_1 z_{2i-1} + b_1 f_\lambda(x_{2i}) = a_1 z_{2i-1} - b_1 \in (-\infty, \lambda], \\
z_{2i} &= a_2 x_{2i} + b_2 f_\lambda(y_{2i}) = a_2 x_{2i} - b_2 \in (-\infty, \lambda].
\end{aligned}$$

Let $m = 2i$, then the proof is complete. The second and last row proof are similar to first row, hence our assertion holds. (iii). Suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty)$. By (4), (5), (7) and induction, then its limiting behavior can be summarized in the following table:

Table 2:

$y_{-1} \in (-\infty, \lambda]$	$z_{-1} \in (A_i^+, B_{i,1}^+]$	$(y_{2i}, z_{2i}) \in (-\infty, \lambda]^2$
$y_{-1} \in (-\infty, \lambda]$	$z_{-1} \in (B_{i,1}^+, C_{i,1}^+]$	$(y_{2i+1}, z_{2i+1}) \in (-\infty, \lambda]^2$
$y_{-1} \in (-\infty, \lambda]$	$z_{-1} \in (C_{i,1}^+, A_{i+1}^+]$	$(y_{2i+1}, z_{2i+1}) \in (-\infty, \lambda]^2$

(iii) is similar to that of (ii) and hence is omitted.

Theorem 1. Suppose $\lambda = 1$. Then a solution $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$ will tend to $(-1, -1, -1)$.

Proof. In view of Lemma 1, we may assume without loss of generality that $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$. For our assumption, we have $a_i\lambda - b_i < \lambda$ for $i = 0, 1, 2$. Furthermore, by (4),

$$\begin{aligned}
x_0 &= a_0 y_{-1} + b_0 f_\lambda(z_{-1}) = a_0 y_{-1} - b_0 \le a_0\lambda - b_0 < \lambda,\\
y_0 &= a_1 z_{-1} + b_1 f_\lambda(x_0) = a_1 z_{-1} - b_1 \le a_1\lambda - b_1 < \lambda,\\
z_0 &= a_2 x_0 + b_2 f_\lambda(y_0) = a_2 x_0 - b_2 \le a_2\lambda - b_2 < \lambda,\\
x_1 &= a_0 y_0 + b_0 f_\lambda(z_0) = a_0 y_0 - b_0 \le a_0\lambda - b_0 < \lambda,\\
y_1 &= a_1 z_0 + b_1 f_\lambda(x_1) = a_1 z_0 - b_1 \le a_1\lambda - b_1 < \lambda,\\
z_1 &= a_2 x_1 + b_2 f_\lambda(y_1) = a_2 x_1 - b_2 \le a_2\lambda - b_2 < \lambda.
\end{aligned}$$

By induction, for any $k \in \mathbb{N}$, we have $(y_k, z_k) \in (-\infty, \lambda]^2$. Hence

$$\begin{aligned}
x_{2k} &= a_0\sigma^k y_{-1} + a_0\sigma^k - 1.\\
x_{2k+1} &= a_0 a_1\sigma^k z_{-1} + a_0 a_1\sigma^k - 1.\\
y_{2k} &= a_1\sigma^k z_{-1} + a_1\sigma^k - 1.\\
y_{2k+1} &= \sigma^{k+1} y_{-1} + \sigma^{k+1} - 1.\\
z_{2k} &= a_0 a_2\sigma^k y_{-1} + a_0 a_2\sigma^k - 1.\\
z_{2k+1} &= \sigma^{k+1} z_{-1} + \sigma^{k+1} - 1.
\end{aligned}$$

Then $\lim_{n\longrightarrow\infty}(x_n, y_n, z_n) = (-1, -1, -1)$. The proof is complete.

Theorem 2. Suppose $\lambda = 1$. Then a solution $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ of (4) with $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$ will tend to $(1, 1, 1)$.

Proof. For our assumption, we have $a_i\lambda + b_i = \lambda$ for $i = 0, 1, 2$. Furthermore, by (4) ,

$$\begin{aligned} x_0 &= a_0 y_{-1} + b_0 f_\lambda(z_{-1}) = a_0 y_{-1} + b_0 > a_0\lambda + b_0 = \lambda, \\ y_0 &= a_1 z_{-1} + b_1 f_\lambda(x_0) = a_1 z_{-1} + b_1 > a_1\lambda + b_1 = \lambda, \\ z_0 &= a_2 x_0 + b_2 f_\lambda(y_0) = a_2 x_0 + b_2 > a_2\lambda + b_2 = \lambda, \\ x_1 &= a_0 y_0 + b_0 f_\lambda(z_0) = a_0 y_0 + b_0 > a_0\lambda + b_0 = \lambda, \\ y_1 &= a_1 z_0 + b_1 f_\lambda(x_1) = a_1 z_0 + b_1 > a_1\lambda + b_1 = \lambda, \\ z_1 &= a_2 x_1 + b_2 f_\lambda(y_1) = a_2 x_1 + b_2 > a_2\lambda + b_2 = \lambda. \end{aligned}$$

By induction, for any $k \in \mathbb{N}$, we have $(y_k, z_k) \in (\lambda, +\infty)^2$. Hence

$$\begin{aligned} x_{2k} &= a_0\sigma^k y_{-1} - a_0\sigma^k + 1. \\ x_{2k+1} &= a_0 a_1 \sigma^k z_{-1} - a_0 a_1 \sigma^k + 1. \\ y_{2k} &= a_1\sigma^k z_{-1} - a_1\sigma^k + 1. \\ y_{2k+1} &= \sigma^{k+1} y_{-1} - \sigma^{k+1} + 1. \\ z_{2k} &= a_0 a_2 \sigma^k y_{-1} - a_0 a_2 \sigma^k + 1. \\ z_{2k+1} &= \sigma^{k+1} z_{-1} - \sigma^{k+1} + 1. \end{aligned}$$

Then $\lim_{n\longrightarrow\infty}(x_n, y_n, z_n) = (1, 1, 1)$. The proof is complete.

The case where $\lambda > 1$.

Lemma 2. Suppose $\lambda > 1$. If $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is a solution of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2$, then there exists an integer $m \in \{-1, 0, \cdots\}$, such that $(y_m, z_m) \in (-\infty, \lambda]^2$.

Proof. (i). Suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$, then we are done. (ii). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda] \cup (-\infty, \lambda] \times (\lambda, +\infty)$. By (4) and induction, the proof is same as (ii), (iii) of Lemma 1. (iii). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$, by (5), (8), (9), let

$$y_{-1} \in \bigcup_{i=0}^{\infty}\{(A_i^-, B_{i,0}^-]\bigcup(B_{i,0}^-, C_{i,2}^-]\bigcup(C_{i,2}^-, A_{i+1}^-]\},$$

$$z_{-1} \in \bigcup_{j=0}^{\infty}\{(A_j^-, B_{j,1}^-]\bigcup(B_{j,1}^-, C_{j,1}^-]\bigcup(C_{j,1}^-, A_{j+1}^-]\}.$$

By (4) and induction, we can get the following table:

Table 3:

y_{-1}	z_{-1}	condition	(y_k, z_k)	
$\in (A_i^-, B_{i,0}^-]$	$\in (A_j^-, B_{j,1}^-]$	$i = j$	$\in (-\infty, \lambda]^2$	$k = 2i$
		$i > j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2j$
		$i < j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2i$
$\in (A_i^-, B_{i,0}^-]$	$\in (B_{j,1}^-, C_{j,1}^-]$	$i \le j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2i$
		$i > j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2j + 1$
$\in (A_i^-, B_{i,0}^-]$	$(C_{j,1}^-, A_{j+1}^-]$	$i \le j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2i$
		$i > j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2j + 1$
$\in (B_{i,0}^-, C_{i,2}^-]$	$\in (A_j^-, B_{j,1}^-]$	$i = j$	$\in (-\infty, \lambda]^2$	$k = 2i$
		$i < j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2i$
		$i > j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2j$
$\in (B_{i,0}^-, C_{i,2}^-]$	$\in (B_{j,1}^-, C_{j,1}^-]$	$i > j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2j + 1$
		$i \le j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2i$
$\in (B_{i,0}^-, C_{i,2}^-]$	$\in (C_{j,1}^-, A_{j+1}^-]$	$i \le j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2i$
		$i > j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2j + 1$
$\in (C_{i,2}^-, A_{i+1}^-]$	$\in (A_j^-, B_{j,1}^-]$	$i = j$	$\in (-\infty, \lambda]^2$	$k = 2i$
		$i < j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2i + 1$
		$i > j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2j$
$\in (C_{i,2}^-, A_{i+1}^-]$	$\in (B_{j,1}^-, C_{j,1}^-]$	$i = j$	$\in (-\infty, \lambda]^2$	$k = 2i$
		$i < j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2i + 1$
		$i > j$	$\in \mathbb{R} \times (-\infty, \lambda]$	$k = 2j + 1$
$\in (C_{i,2}^-, A_{i+1}^-]$	$\in (C_{j,1}^-, A_{j+1}^-]$	$i \le j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2i + 1$
		$i > j$	$\in (\lambda, +\infty) \times (-\infty, \lambda]$	$k = 2j + 1$

For instance, the last row states if $(y_{-1}, z_{-1}) \in (C^-_{i,2}, A^-_{i+1}] \times (C^-_{j,1}, A^-_{j+1}]$.

$$x_0 = a_0 y_{-1} + b_0 \in (\frac{\lambda + a_2\sigma^i - 1}{a_2\sigma^i}, \frac{\lambda + a_1a_2\sigma^i - 1}{a_1a_2\sigma^i}],$$
$$y_0 = a_1 z_{-1} + b_1 \in (\frac{\lambda + a_0\sigma^j - 1}{a_0\sigma^j}, \frac{\lambda + a_0a_2\sigma^j - 1}{a_0a_2\sigma^j}],$$
$$z_0 = a_2 x_0 + b_2 \in (\frac{\lambda + \sigma^i - 1}{\sigma^i}, \frac{\lambda + a_1\sigma^i - 1}{a_1\sigma^i}],$$

$$x_1 = a_0 y_0 + b_0 \in (\frac{\lambda + \sigma^j - 1}{\sigma^j}, \frac{\lambda + a_2\sigma^j - 1}{a_2\sigma^j}],$$
$$y_1 = a_1 z_0 + b_1 \in (C^-_{i-1,2}, A^-_i],$$
$$z_1 = a_2 x_1 + b_2 \in (C^-_{j-1,1}, A^-_j],$$
$$\vdots$$

By induction, if $i \le j$, we can get

$$y_{2i+1} = a_1 z_{2i} + b_1 \in (a_1\lambda + b_1, \lambda] \subseteq (-\infty, \lambda],$$
$$z_{2i+1} = a_2 x_{2i+1} - b_2 \in (a_2\lambda - b_2, \lambda + 2(a_2 - 1)] \subseteq \mathbb{R}.$$

If $i > j$, we can get

$$y_{2j+1} = a_1 z_{2j} + b_1 \in (C^-_{i-j-1,2}, A^-_{i-j}] \subseteq (\lambda, +\infty),$$
$$z_{2j+1} = a_2 x_{2j+1} + b_2 \in (a_2\lambda + b_2, \lambda] \subseteq (-\infty, \lambda].$$

The other rows proof are similar to first row and hence omitted. So by (i), (ii), (iii), conclusion holds. The proof is complete.

Theorem 3. Suppose $\lambda > 1$. Then a solution $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2$ will tend to $(-1, -1, -1)$. In view of Lemma 2, we may assume without loss of generality that $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$. For our assumption, we have $a_i\lambda - b_i < \lambda$ for $i = 0, 1, 2$. So the proof is same as Theorem 1.

The case where $\lambda < -1$.

By arguments similar to those in the lemma 2, we may show the following result.

Lemma 3. Suppose $\lambda < -1$. If $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is a solution of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2$, then there exists an integer $m \in \{-1, 0, \cdots\}$, such that $(y_m, z_m) \in (\lambda, +\infty)^2$.

Proof. For our assumption, we have $a_i\lambda + b_i > a_i\lambda - b_i > \lambda$ for $i = 0, 1, 2$.

(i). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$, then we are done. (ii). Suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty) \cup (\lambda, +\infty) \times (-\infty, \lambda]$. By (4), (10), (11) and induction, we can get the following table:

Table 4:

y_{-1}	z_{-1}	(y_k, z_k)
$y_{-1} \in (A^-_{i+1}, C^-_{i,2}]$	$z_{-1} \in (\lambda, +\infty)$	$(y_{2i+1}, z_{2i+1}) \in (\lambda, +\infty)^2$
$y_{-1} \in (C^-_{i,2}, B^-_{i,0}]$	$z_{-1} \in (\lambda, +\infty)$	$(y_{2i}, z_{2i}) \in (\lambda, +\infty)^2$
$y_{-1} \in (B^-_{i,0}, A^-_i]$	$z_{-1} \in (\lambda, +\infty)$	$(y_{2i}, z_{2i}) \in (\lambda, +\infty)^2$
$y_{-1} \in (\lambda, +\infty)$	$z_{-1} \in (A^-_{i+1}, C^-_{i,1}]$	$(y_{2i+1}, z_{2i+1}) \in (\lambda, +\infty)^2$
$y_{-1} \in (\lambda, +\infty)$	$z_{-1} \in (C^-_{i,1}, B^-_{i,1}]$	$(y_{2i+1}, z_{2i+1}) \in (\lambda, +\infty)^2$
$y_{-1} \in (\lambda, +\infty)$	$z_{-1} \in (B^-_{i,1}, A^-_i]$	$(y_{2i}, z_{2i}) \in (\lambda, +\infty)^2$

For instance, the last row states if $(y_{-1}, z_{-1}) \in (\lambda, +\infty) \times (B^-_{i,1}, A^-_i]$.

$$
\begin{aligned}
x_0 &= a_0 y_{-1} - b_0 > a_0\lambda - b_0 > \lambda, \\
y_0 &= a_1 z_{-1} + b_1 \in (\frac{\lambda - 1 + \sigma^i}{\sigma^i}, \frac{\lambda - 1 + a_0 a_2 \sigma^{i-1}}{a_0 a_2 \sigma^{i-1}}], \\
z_0 &= a_2 x_0 - b_2 > a_2\lambda - b_2 > \lambda, \\
&\vdots \\
y_{2i-1} &= a_1 z_{2i-2} - b_1 \in (\lambda, +\infty), \\
z_{2i-1} &= a_2 x_{2i-1} + b_2 \in (B^-_{0,1}, A^-_0], \\
x_{2i} &= a_0 y_{2i-1} - b_0 \in (\lambda, +\infty), \\
y_{2i} &= a_1 z_{2i} + b_1 \in (\lambda, a_1\lambda + b_1], \\
z_{2i} &= a_2 x_{2i} + b_2 > \lambda.
\end{aligned}
$$

Let $m = 2i$, then the proof is complete. The other rows proof are similar to last row and hence omitted. (iii). Suppose$(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$, by (5), (12), (13), let

$$
(y_{-1}, z_{-1}) \in \bigcup_{i=0}^{\infty}\{(A^+_{i+1}, C^+_{i,2}]\bigcup(C^+_{i,2}, B^-_{i,0}]\bigcup(B^+_{i,0}, A^+_i]\}\times\bigcup_{j=0}^{\infty}\{(A^+_{j+1}, C^+_{j,1}]\bigcup(C^+_{j,1}, B^+_{j,1}]\bigcup(B^+_{j,1}, A^+_j]\}
$$

By (4) and induction, its limiting behavior can be summarized in the following table:

Table 5:

y_{-1}	z_{-1}	condition	(y_k, z_k)	
$\in (A^+_{i+1}, C^+_{i,2}]$	$\in (A^+_{j+1}, C^+_{j,1}]$	$i = j$	$\in (\lambda, +\infty)^2$	$k = 2i + 1$
		$i > j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2j + 1$
		$i < j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2i + 1$
$\in (A^+_{i+1}, C^+_{i,2}]$	$\in (C^+_{j,1}, B^+_{j,1}]$	$i = j$	$\in (\lambda, +\infty)^2$	$k = 2i + 1$
		$i > j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2j + 1$
		$i < j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2i + 1$
$\in (A^+_{i+1}, C^+_{i,2}]$	$(B^+_{j,1}, A^+_j]$	$i \geq j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2j$
		$i < j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2i + 1$
$\in (C^+_{i,2}, B^+_{i,0}]$	$\in (A^+_{j+1}, C^+_{j,1}]$	$i = j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2i$
		$i > j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2j + 1$
		$i < j$	$\in (-\infty, \lambda] \times \mathbb{R}$	$k = 2i$
$\in (C^+_{i,2}, B^+_{i,0}]$	$\in (C^+_{j,1}, B^+_{j,1}]$	$i \leq j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2i$
		$i > j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2j + 1$
$\in (C^+_{i,2}, B^+_{i,0}]$	$\in (B^+_{j,1}, A^+_j]$	$i = j$	$\in (\lambda, +\infty)^2$	$k = 2i$
		$i > j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2j$
		$i < j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2i$
$\in (B^+_{i,0}, A^+_i]$	$\in (A^+_{j+1}, C^+_{j,1}]$	$i = j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2i$
		$i > j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2j$
		$i < j$	$\in (-\infty, \lambda] \times (\lambda, +\infty)$	$k = 2i + 1$
$\in (B^+_{i,0}, A^+_i]$	$\in (C^+_{j,1}, B^+_{j,1}]$	$i \leq j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2i$
		$i > j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2j + 1$
$\in (B^+_{i,0}, A^+_i]$	$\in (B^+_{j,1}, A^+_j]$	$i = j$	$\in (\lambda, +\infty)^2$	$k = 2i$
		$i > j$	$\in (\lambda, +\infty) \times \mathbb{R}$	$k = 2j$
		$i < j$	$\in \mathbb{R} \times (\lambda, +\infty)$	$k = 2i$

For instance, the last row states if $(y_{-1}, z_{-1}) \in (B^+_{i,0}, A^+_i] \times (B^+_{j,1}, A^+_j]$.

$$
\begin{aligned}
x_0 &= a_0 y_{-1} - b_0 \in (\frac{\lambda + 1 - \sigma^i}{\sigma^i}, \frac{\lambda + 1 - a_1 a_2 \sigma^{i-1}}{a_1 a_2 \sigma^{i-1}}], \\
y_0 &= a_1 z_{-1} - b_1 \in (\frac{\lambda + 1 - \sigma^j}{\sigma^j}, \frac{\lambda + 1 - a_0 a_2 \sigma^{j-1}}{a_0 a_2 \sigma^{j-1}}],
\end{aligned}
$$

$$
\begin{aligned}
z_0 &= a_2 x_0 - b_2 \in (\frac{\lambda + 1 - a_0 a_1 \sigma^{i-1}}{a_0 a_1 \sigma^{i-1}}, \frac{\lambda + 1 - a_1 \sigma^{i-1}}{a_1 \sigma^{i-1}}], \\
x_1 &= a_0 y_0 - b_0 \in (\frac{\lambda + 1 - a_1 a_2 \sigma^{j-1}}{a_1 a_2 \sigma^j}, \frac{\lambda + 1 - a_2 \sigma^{j-1}}{a_2 \sigma^{j-1}}], \\
y_1 &= a_1 z_0 - b_1 \in (B^+_{i-1,0}, A^+_{i-1}], \\
z_1 &= a_2 x_1 - b_2 \in (B^+_{j-1,1}, A^+_{j-1}], \\
&\vdots
\end{aligned}
$$

By induction, if $i = j$, we can get

$$
\begin{aligned}
y_{2i-1} &= a_1 z_{2i-2} - b_1 \in (B^+_{0,0}, A^+_0], \\
z_{2i-1} &= a_2 x_{2i-1} - b_2 \in (B^+_{0,1}, A^+_0], \\
x_{2i} &= a_0 y_{2i-1} - b_0 \in (\lambda, a_0 \lambda - b_0], \\
y_{2i} &= a_1 z_{2i-1} + b_1 > \lambda, \\
z_{2i} &= a_2 x_{2i} + b_2 > \lambda.
\end{aligned}
$$

If $i < j$, we can get

$$
\begin{aligned}
y_{2i-1} &= a_1 z_{2i-2} - b_1 \in (B^+_{0,0}, A^+_0], \\
z_{2i-1} &= a_2 x_{2i-1} - b_2 \in (B^+_{j-i,1}, A^+_{j-i}], \\
x_{2i} &= a_0 y_{2i-1} - b_0 \in (\lambda, a_0 \lambda - b_0], \\
y_{2i} &= a_1 z_{2i-1} + b_1 \in \mathbb{R}, \\
z_{2i} &\geq a_2 x_{2i} - b_2 > \lambda.
\end{aligned}
$$

If $i > j$, we can get

$$\begin{aligned} y_{2j-1} &= a_1 z_{2j-2} - b_1 \in (B^+_{i-j,0}, A^+_{i-j}], \\ z_{2j-1} &= a_2 x_{2j-1} - b_2 \in (B^+_{0,1}, A^+_0], \\ x_{2j} &= a_0 y_{2j-1} - b_0 \leq \lambda, \\ y_{2j} &= a_1 z_{2j-1} - b_1 \in (\lambda, a_1\lambda - b_1], \\ z_{2j} &= a_2 x_{2j} + b_2 \in \mathbb{R}. \end{aligned}$$

By (ii), the proof is complete. The other rows proof are similar to last row and hence omitted.

Theorem 4. Suppose $\lambda < -1$. Then a solution $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2$ will tend to $(1, 1, 1)$.

For our assumption, we have $a_i\lambda + b_i > \lambda$ for $i = 0, 1, 2$. In view of Lemma 3, we may assume without loss of generality that $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$. The proof is similar to Theorem 2 and hence omitted.

The case where $\lambda = -1$.

Lemma 4. Suppose $\lambda = -1$. If $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is a solution of (4) with $(y_{-1}, z_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$, then there exists an integer $m \in \{-1, 0, \cdots\}$, such that $(y_m, z_m) \in (\lambda, +\infty)^2$.

Theorem 5. Suppose $\lambda = -1$. suppose $(y_{-1}, z_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$, the every solution of (4) converges to $(1, 1, 1)$.

For our assumption, we have $a_i\lambda + b_i > \lambda$ for $i = 0, 1, 2$. In view of Lemma 4, we may assume without loss of generality that $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$.

Theorem 6. Suppose $\lambda = -1$. suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$. Then any solution of (4) converges to $(-1, -1, -1)$.

The case $\lambda = -1$ is similar to $\lambda = 1$, the proof of Lemma 4, Theorem 5 and Theorem 6, we can refer to Lemma 1, Theorem 1 and Theorem 2.

The case where $-1 < \lambda < 1$.

By arguments similar to those in the Theorem 5 and Theorem 6, we may show the following two results.

Theorem 7. Suppose $-1 < \lambda < 1$, then the following conclusions hold.

(i). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$, the every solution of (4) tend to $(1, 1, 1)$.

(ii). Suppose $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$, the every solution of (4) tend to $(-1, -1, -1)$.

For our assumption, we have $a_i\lambda - b_i < \lambda < a_i\lambda + b_i$ for $i = 0, 1, 2$. So (i) and (ii) respectively are similar Theorem 4 and Theorem 1. Hence the proofs are omitted.

Theorem 8. Suppose $-1 < \lambda < 1$. Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is a solution of (4) with $(y_{-1}z_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda] \cup (-\infty, \lambda] \times (\lambda, +\infty)$. Then its limiting behavior can be summarized in the following table:

Table 6:

(i) $(y_{-1}, z_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda]$

y_{-1}	z_{-1}	condition	$\lim(x_n, y_n, z_n)$
$\in (A_i^+, B_{i,0}^+]$	$\in (A_{j+1}^-, C_{j,1}^-] \cup (C_{j,1}^-, B_{j,1}^-]$ $\cup (B_{j,1}^-, A_j^-]$	$i > j$ $i \leq j$	(1,1,1) (-1,-1,-1)
$\in (B_{i,0}^+, C_{i,2}^+]$	$\in (A_{j+1}^-, C_{j,1}^-] \cup (B_{j,1}^-, A_j^-]$	$i \geq j$ $i < j$	(1,1,1) (-1,-1,-1)
$\in (B_{i,0}^+, C_{i,2}^+]$	$\in (C_{j,1}^-, B_{j,1}^-]$	$i > j$ $i \leq j$	(1,1,1) (-1,-1,-1)
$\in (C_{i,2}^+, A_{i+1}^+]$	$\in (C_{j,1}^-, B_{j,1}^-] \cup (B_{j,1}^-, A_j^-]$	$i \geq j$ $i < j$	(1,1,1) (-1,-1,-1)
$\in (C_{i,2}^+, A_{i+1}^+]$	$\in (A_{j+1}^-, C_{j,1}^-]$	$i > j$ $i \leq j$	(1,1,1) (-1,-1,-1)

Table 7:

(ii) $(y_{-1}, z_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty)$

y_{-1}	z_{-1}	condition	$\lim(x_n, y_n, z_n)$
$\in (A^-_{i+1}, C^-_{i,1}]$	$\in (A^+_j, B^+_{j,0}] \cup (B^+_{j,0}, C^+_{j,2}]$	$i < j$ $i \geq j$	(1,1,1) (-1,-1,-1)
$\in (A^-_{i+1}, C^-_{i,1}]$	$\in (C^+_{j,2}, A^+_{j+1}]$	$i \leq j$ $i > j$	(1,1,1) (-1,-1,-1)
$\in (C^-_{i,1}, B^-_{i,1}]$	$\in (A^+_j, B^+_{j,0}]$	$i < j$ $i \geq j$	(1,1,1) (-1,-1,-1)
$\in (C^-_{i,1}, B^-_{i,1}]$	$\in (B^+_{j,0}, C^+_{j,2}] \cup (C^+_{j,2}, A^+_{j+1}]$	$i > j$ $i \leq j$	(1,1,1) (-1,-1,-1)
$\in (B^-_{i,1}, A^-_i]$	$\in (A^+_j, B^+_{j,0}] \cup (B^+_{j,0}, C^+_{j,2}]$ $\cup (C^+_{j,2}, A^+_{j+1}]$	$i \leq j$ $i > j$	(1,1,1) (-1,-1,-1)

Proof. (i). Suppose $(y_{-1}, z_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda]$, then by (4), (5), (6) and (11) we can get

$$(y_{-1}, z_{-1}) \in \bigcup_{i=0}^{\infty} \{(A^+_i, B^+_{i,0}] \bigcup (B^+_{i,0}, C^+_{i,2}] \bigcup (C^+_{i,2}, A^+_{i+1}]\} \times \bigcup_{j=0}^{\infty} \{(A^-_{j+1}, C^-_{j,1}] \bigcup (C^-_{j,1}, B^-_{j,1}] \bigcup (B^-_{j,1}, A^-_j]\}$$

.

For instance, the last row states if $(y_{-1}, z_{-1}) \in (C^+_{i,2}, A^+_{i+1}] \times (A^-_{j+1}, C^-_{j,1}]$.

$$
\begin{aligned}
x_0 &= a_0 y_{-1} - b_0 \in (\frac{\lambda+1-a_2\sigma^i}{a_2\sigma^i}, \frac{\lambda+1-a_1a_2\sigma^i}{a_1a_2\sigma^i}],\\
y_0 &= a_1 z_{-1} + b_1 \in (\frac{\lambda+a_0a_2\sigma^j-1}{a_0a_2\sigma^j}, \frac{\lambda+a_0\sigma^j-1}{a_0\sigma^j}],\\
z_0 &= a_2 x_0 - b_2 \in (\frac{\lambda+1-\sigma^i}{\sigma^i}, \frac{\lambda+1-a_1\sigma^i}{a_1\sigma^i}],\\
x_1 &= a_0 y_0 + b_0 \in (\frac{\lambda+a_2\sigma^j-1}{a_2\sigma^j}, \frac{\lambda+\sigma^j-1}{\sigma^j}],\\
y_1 &= a_1 z_0 - b_1 \in (C^+_{i-1,2}, A^+_i],\\
z_1 &= a_2 x_1 + b_2 \in (A^-_j, C^-_{j-1,1}],\\
&\vdots
\end{aligned}
$$

By induction, if $i \le j$, we can get

$$
\begin{aligned}
y_{2i+1} &= a_1 z_{2i} - b_1 \in (a_1\lambda - b_1, \lambda] \subseteq (-\infty, \lambda],\\
z_{2i+1} &= a_2 x_{2i+1} - b_2 \le a_2\lambda - b_2 < \lambda.
\end{aligned}
$$

If $i > j$, we can get

$$
\begin{aligned}
y_{2j+1} &= a_1 z_{2j} - b_1 \in (C^+_{i-j-1,0}, A^+_{i-j}] \subseteq (\lambda, +\infty),\\
z_{2j+1} &= a_2 x_{2j+1} + b_2 \in (\lambda, a_2\lambda + b_2) \subseteq (\lambda, +\infty).
\end{aligned}
$$

We may assume without loss of generality that $(y_{-1}, z_{-1}) \in (\lambda, +\infty)^2$ where $i > j$. And assume that $(y_{-1}, z_{-1}) \in (-\infty, \lambda]^2$ where $i \le j$. By Theorem 7, the proof is complete. The other rows proof are similar to last row and hence omitted. The conclusion (ii) is similar to (i) and proof is omitted.

3 Discussion

The result in the previous section for the system (4) can easily be translated into result for (3). We summa as follow:

(1) Suppose $\lambda = 1$. A solution $\{u_n\}_{n=0}^{\infty}$ of (3) with $(u_{-2}, u_{-1}) \in (\lambda, +\infty)^2$ will tend towards 1. If $(u_{-2}, u_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$, the solutions will tend towards -1.

(2) Suppose $\lambda = -1$. A solution $\{u_n\}_{n=0}^{\infty}$ of (3) with $(u_{-2}, u_{-1}) \in (-\infty, \lambda]^2$ will tend towards -1. If $(u_{-2}, u_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$, the solutions will tend towards 1.

(3) Suppose $-1 < \lambda < 1$. A solution $\{x_n\}_{n=0}^{\infty}$ of (3) with $(u_{-2}, u_{-1}) \in (\lambda, +\infty]^2$ will tend towards 1. If $(u_{-2}, u_{-1}) \in (-\infty, \lambda]^2$, the solutions will tend towards -1. If $(u_{-2}, u_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2/(\lambda, +\infty)^2$, results in detail please see Theorem 8.

(4) Suppose $\lambda < -1$. A solution $\{u_n\}_{n=0}^{\infty}$ of (3) with $(u_{-2}, u_{-1}) \in \mathbb{R}^2$ will eventually fall into $(\lambda, +\infty)^2$ and approach 1.

(5) Suppose $\lambda > 1$. A solution $\{u_n\}_{n=0}^{\infty}$ of (3) with $(u_{-2}, u_{-1}) \in \mathbb{R}^2$ will eventually fall into $(-\infty, \lambda]^2$ and approach -1.

In neural network terminologies, we have discussed a simple neuron recurrent McCulloch-Pitts-type neural network with a threshold and 3-periodic coefficients. Such an observation seems to appear in many natural processes and hence our model may be use to explain such phenomena. It is also expected that when a group of neural units interact with each other in a network where each unit is governed by evolutionary laws of the form (3), complex but manageable analytical results can be obtained. These will be left to other studiers in the future.

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