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Hidden Structure and Complex Dynamics of Hyperchaotic Attractors

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Abstract: The paper introduces new 4-D dynamical systems ensuring full hyperchaotic patterns. Its focal statement appears in the novelty of the equation's specification of both models. Indeed, the two built systems integrating small set of nonlinear terms are not expanded variants of 3-D nonlinear systems into the fourth dimension. To explore the basic behavior of the models, we display the phase portraits of the related hyperchaotic attractors projected onto the 3-D representation spaces. The simulations exhibit hyperchaotic attractors with wings and scrolls. A collection of Poincaré Maps reveals the intricate and elegant structure of the first one. The computation of the Lyapunov exponents establishes presence of hyperchaos since two positive exponents are found. Indexes of stability of the equilibrium points corresponding to both systems are also examined. Besides, we present bifurcation diagrams highlighting the arrangement of hyperchaotic bubbles and periodic windows for a restricted range of the control parameter, for the following system. Eventually, the two systems specify dissimilar hyperchaos patterns. The phase portraits of the first model hide a very specific composition unveiled by the Poincaré maps. On the other hand, the phase portraits of the following system reveal distinct contours and uncover rich sensitivity to the control parameter.

Keywords: 4-D system; Hyperchaos; Phase Portraits; Poincaré Map; Diagrams of bifurcation.

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1 Introduction

Hyperchaos concept was firstly introduced in the seminal paper of Rössler [1] to assert the patterns of dynamical systems when more than one positive Lyapunov exponent is found [2]. Some of discovered hyperchaotic systems are thereafter established mainly by the extension of well-known 3-D chaotic to the fourth dimension. These hyperchaotic applications start from a fully 3-D chaotic system adding a state feedback controller [see for example, 3-8]. However, such 4-D models are only expanded editions in the phase hyperspace.

Purposefully creating hyperchaos can be a nontrivial task to focus a new kind of dynamical patterns as shown by several 4-D models recently discovered [9-14, and references therein]. This paper introduces two hyperchaotic systems not derived from previous 3-D models since we do not apply hyperchaotification techniques. At best of our knowledge, the presented intentionally constructed 4-D hyperchaotic attractors have silhouette and topology distinct from previous models. Besides, the detection of new 4-D hyperchaotic attractors extends our knowledge of disordered system, the outcome of the feedbacks, and the hyper-symmetry arrangements.

Section 2 investigates the basic characteristics of the introduced autonomous four-dimensional systems of first order differential equations. Indeed, the systems integrate a common block of two equations constituted by a deeply modified 2-D Lotka-Volterra oscillator [15-16]. We display the projected phase portrait into the three-dimensional spaces and point chiefly to the intricate structure and/or patterns of the hyperchaos. Poincaré maps and bifurcation diagrams are computed to report these results.

Concluding remarks report the singularity of the models and the variety of the dynamical behavior displayed. The systems provide novel contribution to the hyperchaos literature since all equations of each model contains at most 3-term cross products.

2 The Hyperchaotic 4-D Models

To formulate models emitting hyperchaotic signals, numerous techniques can be applied [17-19]. In this paper, we selected a basic sub-system as the core of the systems. It is constituted by a modified version of the well-established 2-D Lotka-Volterra oscillator.

Indeed, the core of the models is:

$$\begin{cases} x' = x\,(1-y) \\ \\ y' = (x^2-1)\,y \end{cases}$$

where x and y, the state variables. This oscillatory mechanism was connected previously to a feedback equation to determine three distinct 3-D chaotic models with different topology of their strange attractors [20-22]. In the following two sections, we present the two new hyperchaotic systems with the same technique of connection.

2.1 A hyperchaotic attractor with hidden structure

The first system is governed by four nonlinear differential equations:

$$\begin{cases} x' = x\,(1-y) + \alpha z \\ y' = \beta\,(x^2-1)\,y \\ z' = \gamma\,(1-y)\,v \\ v' = \eta z \end{cases} \tag{1}$$

where $x, y, z,$ and v are the state variables of the model, and α, β, γ and η real parameters. Equations embed only eight terms on the right-hand side, three of them are nonlinear, i.e. two quadratic terms xy, and vy, and a unique cubic cross-term x^2y.

One would expect the emergence of hyperchaotic patterns in our application. We notice that the system is not derived from a previous model. Indeed, it is not an expanded version of a 3-D chaotic system adding a state feedback loop formulated within a supplementary equation. Besides, any configuration of its sub-systems carried out by three state variables amongst the four is not sustainable.

2.1.1 The stability attributes

The exploration of the widest dynamical behaviors leads us to select specific values of the parameters between several specifications to simplify the analysis.

Let $P_0\,(\alpha, \beta, \gamma, \eta) = (-2, 1, 0.2, 1)$, the equilibria of the system (1) are found by setting the expression $x' = y' = z' = v' = 0$, articulated as follow:

$$\begin{cases} x\,(1-y)+\alpha z=0 \\ \beta\,(x^2-1)\,y=0 \\ \gamma\,(1-y)\,v=0 \\ \eta z=0 \end{cases}$$

The coordinates of the equilibria are the origin $S_0 : (0,0,0,0)$ and two sets of parametric solutions: $S_1 : (1,1,0,v)$ and $S_2 : (-1,1,0,v)$. Eigenvalues λ_i and the stability features of the related solutions are determined from the characteristic equation $|J-\lambda I|=0$, where I is the unit matrix and J is the Jacobian matrix of the model:

$$J=\begin{pmatrix} 1-y & -x & -2 & 0 \\ 2xy & x^2-1 & 0 & 0 \\ 0 & -0.2v & 0 & 0.2\,(1-y) \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The volume contraction of the flot is given by: $\nabla V = tr\,(J) = x^2-y$. However, the dissipativity in the phase hyperspace is enclosed in the domain:$x^2-y<0$, and thus the system is a dissipative system. In this domain, the orbits converge to a specific subset of zero volume as $t\to\infty$ exponentially; i.e. $dV/dt=\exp(x^2-y)$. For P_0, all solutions are unstable and distinguished with different indexes of stability[1]. We have the following cases:

1. For $S_0(0,0,0,0)$, $(Det(J_0)=0.2>0)$ the corresponding characteristic equation is $(\lambda^2-1)(\lambda^2-0.2)=0$, that is $\lambda_1=-1, \lambda_2=-0.2, \lambda_3=0.2, \lambda_4=1$ and it is a saddle focus because it has Index-2.
2. For the line $S_1(1,1,0,v)_{v\in\mathbb{R}}$, $(Det(J_1)=0)$ and we have for $v\geq 0, Re(\lambda_1)>0, Re(\lambda_2,\lambda_3,\lambda_4)\leq 0$ and is a saddle points because it has Index-1 and for $v<0$, $Re(\lambda_1,\lambda_2)>0, Re(\lambda_3,\lambda_4)\leq 0$ and it is a saddle foci because it has Index-2.
3. For the line $S_2(-1,1,0,v)_{v\in\mathbb{R}}$, $(Det(J_2)=0)$ and we have for $v\geq 0, Re(\lambda_1,\lambda_2)>0, Re(\lambda_3,\lambda_4)\leq 0$ and it is a saddle foci because it has Index-2 and for $v\leq 0$, $Re(\lambda_1)>0, Re(\lambda_2,\lambda_3,\lambda_4)\leq 0$ and it is a saddle points because it has Index-1.

[1] Index reports the number of eigenvalues with real parts $Re\lambda>0$. From 1 to 4, it indicates the degree of instability. Index-0: null or negative real parts of all eigenvalues of the equilibrium characterize its stability.

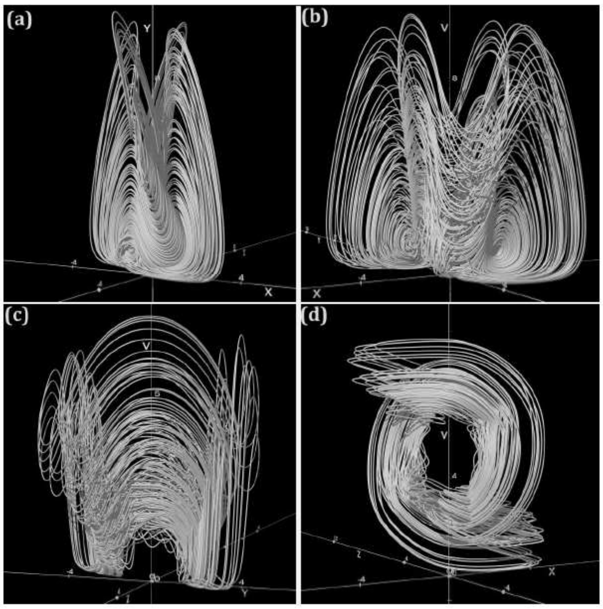

Figure 1: Projections of the 4-D hyperchaotic system into the 3-D phase portraits. (a) Projection on $x-y-z$ space, (b) Projection on $x-y-v$ space, (c) Projection on $z-y-v$ space, and (d) Projection on $z-v-x$ space

2.1.2 Global Behavior and Patterns

To substitute the unfeasibility of a four-dimensional representation, four projections in the 3-D phase portraits are exhibited, respectively the $x-y-z$, the $x-y-v$, the $z-y-v$, and $z-v-x$ spaces (Fig. 1). The orbits have intricate paths following butterfly wings and scrolls. The trajectories are extremely abundant and dense taking relatively regular and simple forms. However, does the new system have explicitly and globally a hyperchaotic nature?

It is known that the spectrum of Lyapunov exponents is the most useful diagnostic to quantify chaos. When the nearby trajectories in the phase space diverge at exponential rates, giving positive Lyapunov exponents, the dynamics become unpredictable. Any system containing at least two positive Lyapunov exponents is defined to be hyperchaotic. The computed Lyapunov exponents LE$_i arethefollows: \ LE_1 = 1.85, LE_2 = 0.34, LE_3 \approx$

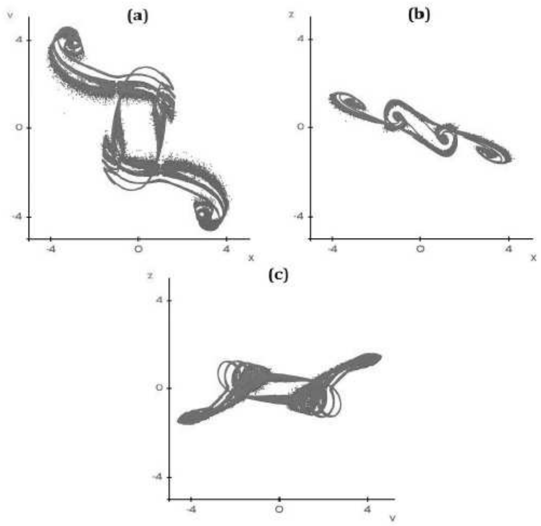

Figure 2: Poincaré maps of the System (1) for $y = 2$. (a) Projection on $v - z$ of the Poincaré map, (b) Projection on $x - z$ of the Poincaré map, and (c) Projection on $x - v$ of the Poincaré map.

$0, LE_4 = -34.2$.The system (1) can be classified hyperchaotic since:

$$\begin{cases} LE_1 > LE_2 > 0, LE_3 = 0, LE_4 < 0, \\ LE_1 + LE_2 + LE_4 < 0, \end{cases}$$

The Kaplan-Yorke dimension of the attractor reaches $D \approx 3.064$. To describe the folding properties of chaos, we apply the technique of the Poincaré map. Take

$$\Sigma = \{(x, y, z, v) \in \mathbb{R}^4 : y = 2\}$$

as a crossing section. We select the mapping on Σ from the 4-D space not to the 3-D phase spaces but on 2-D surfaces.

The associated Poincaré map on the three phase planes show that the system has extremely rich, intricate and elegant structure (Fig. 2). Wraps, branches and twigs unravel the complex folding of the hyperchaotic attractor. In fact, such mapping can exhibit nine other 2-D maps for x, z and v as crossing sections. Similar dynamical structures are also found indicating complex envelops of the orbits.

2.2 A hyperchaotic Attractor with typical wings

The successive hyperchaotic system embeds also the same variant of the well-established 2-D Lotka-Volterra oscillator as the core of the model. The four nonlinear differential equations governing the new system are:

$$\begin{cases} x' = x\,(1-y) + \alpha z \\ y' = \beta\,(x^2-1)\,y \\ z' = \varphi x + (1-y)\,z + sv \\ v' = \psi xy \end{cases} \tag{2}$$

where x, y, z, and v the state variables of the model, and $\alpha, \beta, \varphi, s$, and ψ real parameters. Equations embed four nonlinear terms, three of them are quadratic, i.e. two xy, and yz, and a unique cubic cross-term; i.e. x^2y. We noticed also that the system is not derived from previous 4-D or 3-D chaotic models.

2.2.1 The equilibrium characteristics

Let the set of parameters $C\,(\alpha, \beta, \varphi, s, \psi) = (1, -0.7, -0.1, 1, -0.2)$, the exploration of the system leads us to determine the equilibria of the system. These points are found by setting: $x' = y' = z' = v' = 0$. We obtain the coordinates of three equilibria: the origin $S_0(0,0,0,0)$, $S_1(1,0,1,1.1)$, and $S_2(-1,0,-1,-1.1)$. The eigenvalues λ_i and the stability features of the related solutions are determined from the characteristic equation $|J - \lambda I| = 0$, where I is the unit matrix and J is the Jacobian matrix of the model. We have the following cases:

1. For $S_0(0,0,0,0)$, the corresponding characteristic equation is $\lambda(0.7-\lambda)(\lambda^2+2\lambda+1.1) = 0$, that is $\lambda_1 = 0, \lambda_2 = 0.7, \lambda_3 = -1-0.31i, \lambda_4 = -1+0.31i$ and it is a saddle points because it has Index-1.

2. For $S_1(1,0,1,1.1)$, the corresponding characteristic equation is $\lambda^2(\lambda^2+2\lambda+0.9) = 0$, that is $\lambda_1 = 0, \lambda_2 = -1.31, \lambda_3 = -0.68$ and it has Index-0.

3. For $S_2(-1,0,-1,-1.1)$, the corresponding characteristic equation is $\lambda^2(\lambda^2+2\lambda+1.1) = 0$, that is $\lambda_1 = 0, \lambda_2 = -1-0.3i, \lambda_3 = -1+0.31i$ and it has Index-0.

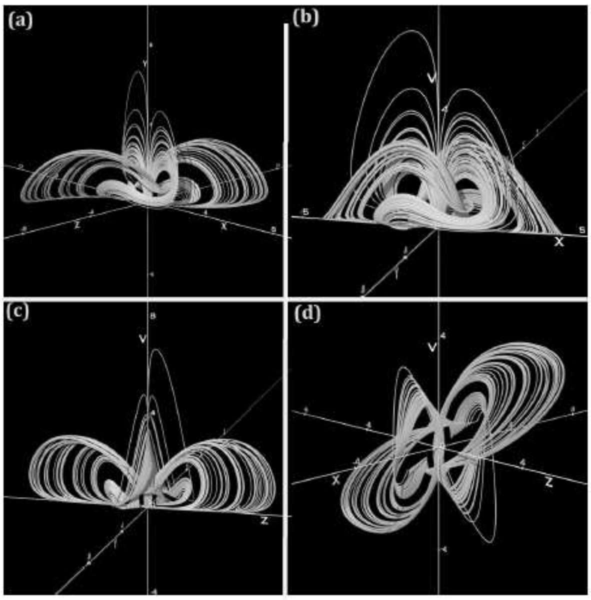

Figure 3: 4-D Attractor for the C parameters. (a) Projection on $z-y-x$ space, (b) Projection on $v-y-z$ space, (c) Projection on $v-y-x$ space, and (d) Projection on $x-v-z$ space.

With the initial conditions $Ic(0.1, 0.1, 0.1, 0.1)$, a 4-D hyperchaotic attractor appears throughout projections into the four 3-D phase spaces (Fig. 3). The system exhibits well-defined disposition of several wings and scrolls, substantiating the novelty of the attractor.

2.2.2 Global Behavior of the hyperchaotic System

The computation of the diagrams of bifurcation could emphasize the range of periodic and non-periodic dynamics. The numerical analysis started with the initial conditions $Ic(0.1, 0.1, 0.1, 0.1)$ show extremely rich and intricate bifurcations (Fig. 4). Chaos bubbles and periodic windows indicate the complex envelops of the dynamics.

3 Concluding Remarks

The introduced 4-D systems display different and noticeable attributes. Embedding a modified 2-D Lotka-Volterra oscillator, both systems depicted complex scroll butterfly-

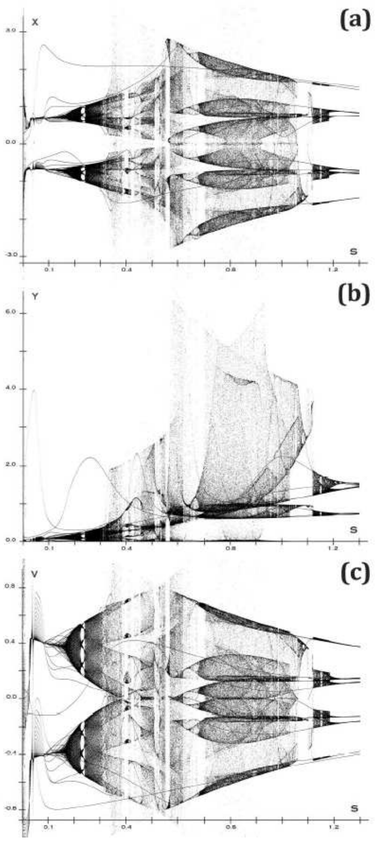

Figure 4: Diagrams of bifurcation for $z = 0$ and the control parameter $s \in]0, 1.3[$. (a) Bifurcation diagram of x, (b) Bifurcation diagram of y, and (c) Bifurcation diagram of v.

shaped attractors with perceptible contours. The first model is topologically compact but its elegant structures are hidden and found by mapping the Poincaré sections. The second model displaying plain hyperchaotic attractor, exhibits wing symmetries and magnified scrolls. The prevailing feature of the systems emerges in the independency of the hyper attractors from previous systems. These intentionally constructed 4-D hyperchaotic models do not reincarnate known hyperchaotic patterns. The appearance and also the characteristics of the new attractors are utterly distinguished from the other existing hyperchaotic systems (4-D Lorenz–Haken system, 4-D hyperchaotic Chen, 4-D hyperchaotic Chua's circuit, etc.). On the other hand, several specifications of parameters sets have been experimented to opt for the simplest ones to induce the widest range of dynamical behaviors. The new systems could be immediately suitable for digital signal encryption in the communication field providing a very large set of encryption keys. Eventually, the exploration of hyperspace dynamics could enhance our understanding of the science of process linking ordered and disordered sequences. **Acknowledgments**: The author

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