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Limits of Solutions of a Recurrence Relation with Bang Bang Control

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Abstract: In this paper, we consider a three term nonlinear recurrence $x_n = ax_{n-2} + bH(x_{n-1}) + c$ where $a > 0$ and b, c are real numbers and H is the Heaviside step function. We are able to derive the exact relations between the initial values x_{-2} and x_{-1} with the limiting behaviors of the solution determined by them. In particular, when $a \in (0, 1)$ and $b > 0$, we are able to show that all solutions $\{x_k\}_{k=-2}^{\infty}$ converge if, and only if, (i) $(c + b)/(1 - a) < 0$, (ii) $(c - b)/(1 - a) > 0$, (iii) $(c + b)/(1 - a) = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^-$; or, (iv) $(c - b)/(1 - a) = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^+$.

Keywords: Three term nonlinear recurrence, Heaviside function, limiting behavior.

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1 Introduction

Three term recurrence relations of the form

$$y_n = F(y_{n-1}, y_{n-2}),\ n \in \mathbf{N} = \{0, 1, 2, 3, ...\},$$

arise in many studies of natural phenomena. A well known example is the relation

$$y_n = y_{n-1} + y_{n-2},\ n \in \mathbf{N},$$

which is satisfied by the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$. When F is a continuous function in the above recurrence relation, there are now numerous studies, but when F is discontinuous, relatively few studies are available (see e.g. [1–4]). However, (discontinuous) on-off control functions such as

$$H(u) = \begin{cases} 1, & u \le 0 \\ -1, & u > 0 \end{cases},\ G(u) = \begin{cases} 1, & u \ge 0 \\ -1, & u < 0 \end{cases}, \tag{1}$$

or

$$H_\lambda(u) = \begin{cases} 1, & u \le \lambda \\ -1, & u > \lambda \end{cases},\ \lambda \in \mathbf{R}, \tag{2}$$

etc. are common and therefore it is of great importance to consider prototype models and study their properties.

In this paper, we consider the following recurrence relation

$$y_n = ay_{n-2} + bH_\lambda(y_{n-1}) + c,\ n \in \mathbf{N}, \tag{3}$$

where $a \in (0, +\infty), b, c \in \mathbf{R}$ and $H_\lambda : \mathbf{R} \to \mathbf{R}$ is the bang bang function defined by (2). Clearly, given any initial pair (y_{-2}, y_{-1}) in $\mathbf{R}^2$, we can generate through (3) a unique real sequence $\{y_n\}_{n=-2}^{\infty}$. Such a sequence is called a solution of (3) originated from (y_{-2}, y_{-1}).

There are many qualitative properties of this nonlinear recurrence which are worthy of studying. Here, we are interested in the limit of the solution sequence $\{y_n\}_{n=-2}^{\infty}$ originated from $\mathbf{R}^2$.

As we will see below, there are only a few types of limiting behaviors for solutions of (3) and we can also determine exactly the 'initial region' from which each type of solutions originate from.

Since there are four real parameters in the nonlinear model (3), the above precise information may seem difficult. Fortunately, we may resort to linear recurrences and transformations for help.

Indeed, let $\{y_k\}_{k=-2}^{\infty}$ be real sequences that satisfy

$$y_{2k} = ay_{2k-2} + d,\ k \in \mathbf{N}, \tag{4}$$

$$y_{2k+1} = ay_{2k-1} + d,\ k \in \mathbf{N}, \tag{5}$$

where $a \in (0, +\infty)$ and d is a real number. Then the following facts are easily obtained by induction.

- If $a \neq 1$ and $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (4), then

$$y_{2k} = a^{k+1} y_{-2} + \frac{(1-a^{k+1})}{1-a} d, \ k \in \mathbf{N}. \tag{6}$$

- If $a \neq 1$ and $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (5), then

$$y_{2k+1} = a^{k+1} y_{-1} + \frac{(1-a^{k+1})}{1-a} d, \ k \in \mathbf{N}. \tag{7}$$

- If $a = 1$ and $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (4), then

$$y_{2k} = y_{-2} + (k+1)d, \ k \in \mathbf{N}. \tag{8}$$

- If $a = 1$ and $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (5), then

$$y_{2k+1} = y_{-1} + (k+1)d, \ k \in \mathbf{N}. \tag{9}$$

Therefore, if $b = 0$ in (3), then it reduces to the linear recurrence relation

$$y_n = a y_{n-2} + c, \ n \in \mathbf{N}, \tag{10}$$

and its asymptotic behavior is quite trivial. Indeed, suppose $a \in (0,1)$. Then any solution $\{y_n\}_{n=-2}^{\infty}$ of (10) satisfies (4) and (5) for $d = c$. Hence by (6) and (7), any solution of (10) tends to $c/(1-a)$. Suppose $a = 1$. Then any solution $\{y_n\}_{n=-2}^{\infty}$ of (10) satisfies (4) and (5) for $d = c$. Hence by (8) and (9), we may see that for any solution $\{y_n\}_{n=-2}^{\infty}$ of (10), $\lim_n y_n = +\infty$ when $c > 0$, $\lim_n y_n = -\infty$ when $c < 0$ while $y_{2n} = y_{-2}$ and $y_{2n+1} = y_{-1}$ for $n \in \mathbf{N}$ when $c = 0$.

Suppose $a > 1$. Then a solution $\{y_n\}_{n=-2}^{\infty}$ of (10) satisfies (4) and (5) for $d = c$. Hence by (6) and (7), we may summarize its limiting behavior in the following table:

Table 1:

y_{-2}	y_{-1}	y_{2n}	y_{2n+1}
$= c/(1-a)$	$= c/(1-a)$	$\to c/(1-a)$	$\to c/(1-a)$
$= c/(1-a)$	$> c/(1-a)$	$\to c/(1-a)$	$\to +\infty$
$> c/(1-a)$	$= c/(1-a)$	$\to +\infty$	$\to c/(1-a)$
$= c/(1-a)$	$< c/(1-a)$	$\to c/(1-a)$	$\to -\infty$
$< c/(1-a)$	$= c/(1-a)$	$\to -\infty$	$\to c/(1-a)$
$> c/(1-a)$	$> c/(1-a)$	$\to +\infty$	$\to +\infty$
$< c/(1-a)$	$< c/(1-a)$	$\to -\infty$	$\to -\infty$
$> c/(1-a)$	$< c/(1-a)$	$\to +\infty$	$\to -\infty$
$< c/(1-a)$	$> c/(1-a)$	$\to -\infty$	$\to +\infty$

For instance, the second row states that if $y_{-2} = y_{-1} = c/(1-a)$, then $y_n \to c/(1-a)$; while the last row states that if $y_{-2} < c/(1-a)$ and $y_{-1} > c/(1-a)$, then $\lim_n y_{2n} = -\infty$ and $\lim_n y_{2n+1} = +\infty$. We remark that the condition $y_{-2} = y_{-1} = c/(1-a)$, as can be seen from (4) and (5), actually implies $y_n = c/(1-a)$ for $n \in \mathbf{N}$. However, in this paper, we only emphasize on limits and hence similar remarks in later discussions will be skipped.

Next, we assume that $b \neq 0$. Then by the transformation $x_n = y_n - \lambda$, we see that (3) is equivalent to

$$x_n = ax_{n-2} + bH(x_{n-1}) + (c + (a-1)\lambda),\ n \in \mathbf{N}, \tag{11}$$

where H is the Heaviside function defined in (1). In particular, if $a = 1$ and $b \neq 0$, then (11) is reduced to

$$x_n = x_{n-2} + bH(x_{n-1}) + c,\ n \in \mathbf{N}. \tag{12}$$

Furthermore, if $a \in (0,1)$ and $b > 0$, or, $a > 1$ and $b < 0$, then by the transformation $z_n = \frac{1-a}{b}x_n$, (11) is equivalent to

$$z_n = az_{n-2} + (1-a)H(z_{n-1}) + d,\ n \in \mathbf{N}; \tag{13}$$

while if $a \in (0,1)$ and $b < 0$, or, $a > 1$ and $b > 0$, then by the same transformation $z_n = \frac{1-a}{b}x_n$, (11) is equivalent to

$$z_n = az_{n-2} + (1-a)G(z_{n-1}) + d,\ n \in \mathbf{N}, \tag{14}$$

where G is the Heaviside function defined in (1) and

$$d = \frac{1-a}{b}(c + (a-1)\lambda).$$

Therefore, we may turn our attention to the equations (12), (13) and (14). Since (14) is similar to (13), we may further turn our attention to the following equation

$$x_n = ax_{n-2} + bH(x_{n-1}) + c,\ n \in \mathbf{N}, \tag{15}$$

which includes (12) and (13) by assuming the cases: (i) $a = 1, b > 0$, (ii) $a = 1, b < 0$, (iii) $a \in (0,1), b > 0$, or (iv) $a > 1, b < 0$.

Henceforth, we will discuss the limiting behaviors of solutions of (15) under the four different sets of conditions on a and b.

To state the corresponding results, it is convenient to introduce some notations. First we set

$$\begin{aligned}
\alpha_{\pm} &= \frac{c \pm b}{1-a}, \\
a^{\pm}_{\eta,k} &= \frac{1}{a^k}\left(\eta - \frac{1-a^k}{1-a}(c \pm b)\right),\ k \in \mathbf{N}, \\
a^{\pm}_{\eta,-k} &= a^k\eta + \frac{1-a^k}{1-a}(c \pm b),\ k \in \mathbf{N}, \\
b^{\pm}_k &= -k(c \pm b),\ k \in \mathbf{N}.
\end{aligned}$$

We will also set

$$a_k^{\pm} = a_{0,k}^{\pm}, a_{-k}^{\pm} = a_{0,-k}^{\pm}, k \in \mathbf{N}.$$

Next, if I and J are real intervals, their cross product $I \times J$ will be denoted by IJ, and we will assume that this product receives the **priority** attention in a mathematical expression. For instance, if we set

$$\mathbf{R}^- = (-\infty, 0], \mathbf{R}^+ = (0, \infty),$$

then $\{\mathbf{R}^+\mathbf{R}^+, \mathbf{R}^+\mathbf{R}^-, \mathbf{R}^-\mathbf{R}^+, \mathbf{R}^-\mathbf{R}^-\}$ is a partition of $\mathbf{R}^2$. Other subsets of the plane will be introduced in the subsequent sections. Here we will employ the following notations

$$a + \Omega = \{a + x |\ x \in \Omega\}$$

and

$$a\Omega = \{ax |\ x \in \Omega\}$$

for any $\Omega \subseteq \mathbf{R}$ and $a \in \mathbf{R}$.

2 The Case where $a = 1$ and $b > 0$

Under the assumption that $a = 1$ and $b > 0$, the limiting behavior of (15) only depends on $c - b$ and $c + b$. Since $c - b < c + b$, we need to consider five cases (i) $0 < c - b$, (ii) $c + b < 0$, (iii) $c + b = 0$, (iv) $c - b = 0$, and (v) $c - b < 0 < c + b$.

Theorem 2.1. Suppose $a = 1, b > 0$ and $0 < c - b$. Then every solution of (15) tends to $+\infty$.

Proof. Let $\{x_k\}_{k=-2}^{\infty}$ be a solution of (15). We first show that exists $m \geq -2$ such that $x_m \in \mathbf{R}^+$. Indeed, suppose to the contrary that $x_k \leq 0$ for all $k \geq -2$. Then $x_k = x_{k-2} + b + c$ for all $k \in \mathbf{N}$. One sees immediately from (8) and (9) $\lim_{n\to\infty} x_n = +\infty$, which is a contradiction.

Next we assert that there exists $m \geq -2$ such that $x_m, x_{m+1} \in \mathbf{R}^+$. Indeed, by the previous discussion, there is $m_0 \geq -2$ such that $x_{m_0} \in \mathbf{R}^+$. If $x_{m_0+1} \in \mathbf{R}^+$, we are done. Otherwise, from

$$x_{m_0+2} = x_{m_0} - b + c > -b + c > 0$$

we have $x_{m_0+2} \in \mathbf{R}^+$. Repeating the argument, we either find $m > m_0$ such that $x_m, x_{m+1} \in \mathbf{R}^+$ or one has that the subsequence x_{m_0+2k} lies in $\mathbf{R}^+$ whereas $x_{m_0+2k+1} \notin \mathbf{R}^+$. This shows that the subsequence $\{x_{m_0+2k+1}\}$ satisfies equation (4) or (5) for $d = -b + c$ and hence $\lim x_{m_0+2k+1} = +\infty > 0$, a contradiction.

Therefore, we may suppose without loss of generality that $x_{-2}, x_{-1} \in \mathbf{R}^+$. Then by (15) and induction, $x_n \in \mathbf{R}^+$ for all $n \geq -2$. Thus $x_n = x_{n-2} - b + c$ for $n \in \mathbf{N}$. In view of (8) and (9), $\lim_{n\to\infty} x_n = +\infty$. The proof is complete.

Theorem 2.2. Suppose $a = 1, b > 0$ and $c + b < 0$. Then every solution of (15) tends to $-\infty$.

The proof is similar to that of Theorem 2.1 and hence skipped.

In the next result, we assume that $a = 1, b > 0$ and $c + b = 0$. Then $0 = b_0^- < b_1^- < \cdots < b_k^- \to +\infty$. If we let

$$C^{(k)} = (b_k^-, b_{k+1}^-],\ \ k \in \mathbf{N},$$

then

$$C^{(0)} - b + c \subseteq \mathbf{R}^-,$$

$$C^{(k)} - b + c = C^{(k-1)},\ \ k \geq 1$$

and

$$\mathbf{R}^+ = \bigcup_{k=0}^{\infty} C^{(k)}.$$

Theorem 2.3. Suppose $a = 1, b > 0$ and $c + b = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behavior can be summarized in the following table:

Table 2:

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$\in \mathbf{R}^-$	$\in \mathbf{R}^-$		$\to x_{-2}$	$\to x_{-1}$
$\in \mathbf{R}^+$	$\in \mathbf{R}^-$		$\to x_{-2}$	$\to -\infty$
$\in \mathbf{R}^-$	$\in \mathbf{R}^+$		$\to -\infty$	$\to x_{-1}$
$\in C^{(k)}$	$\in C^{(s)}$	$0 \leq s < k$	$\to x_{2s}$	$\to -\infty$
$\in C^{(k)}$	$\in C^{(s)}$	$0 \leq k \leq s$	$\to -\infty$	$\to x_{2k-1}$

Proof. (i) Suppose $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$. Then from (15) we see that

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} + b + c = x_{-2} \in \mathbf{R}^-, \\ x_1 &= x_{-1} + bH(x_0) - b = x_{-1} + b + c = x_{-1} \in \mathbf{R}^-. \end{aligned}$$

By induction, one may easily see that $x_{2n} = x_{-2}$ and $x_{2n+1} = x_{-1}$ for all $n \in \mathbf{N}$.

(ii) Suppose $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then by (15),

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} + b + c = x_{-2} \in \mathbf{R}^+, \\ x_1 &= x_{-1} + bH(x_0) + c = x_{-1} - b + c < x_{-1} \in \mathbf{R}^-. \end{aligned}$$

By induction, $x_{2n} = x_{-2} \in \mathbf{R}^+$ and $x_{2n+1} \in \mathbf{R}^-$ for all $n \in \mathbf{N}$. Furthermore, by (5) and (9) with $d = -b + c = -2b$, $\lim_n x_{2n+1} = -\infty$ as required.

(iii) As in (ii), we may show in similar manners that $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^+$ implies $\lim x_{2n} = -\infty$ and $\lim x_{2n+1} = x_{-1}$.

(iv) Suppose $(x_{-2}, x_{-1}) \in C^{(k)}C^{(s)}$ where $0 \le s < k$. Then by (15),

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} - b + c \in C^{(k)} - b + c = C^{(k-1)}, \\ x_1 &= x_{-1} + bH(x_0) + c = x_{-1} - b + c \in C^{(s)} - b + c = C^{(s-1)}, \end{aligned}$$

and by induction, $(x_{2s}, x_{2s+1}) \in \mathbf{R}^+\mathbf{R}^-$. We may proceed as in (ii) to obtain our conclusion.

(v) Suppose $(x_{-2}, x_{-1}) \in C^{(k)}C^{(s)}$ where $0 \le k \le s$. Then by (15), if $k = 0$, then

$$x_0 = x_{-2} + bH(x_{-1}) + c = x_{-2} - b + c \in C^{(0)} - b + c \subseteq \mathbf{R}^-.$$

That is $(x_{-1}, x_0) \in \mathbf{R}^+\mathbf{R}^-$. If $k > 0$, then

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} - b + c \in C^{(k)} - b + c = C^{(k-1)}, \\ x_1 &= x_{-1} + bH(x_0) + c = ax_{-1} - b + c \in C^{(s)} - b + c = C^{(s-1)}, \end{aligned}$$

and by induction, $(x_{2k-1}, x_{2k}) \in \mathbf{R}^+\mathbf{R}^-$. We may now proceed as in (ii) to obtain our conclusion.

The proof is complete.

In the next result, we assume that $a = 1, b > 0$ and $c - b = 0 < c + b$. Then $0 = b_0^+ > b_1^+ > \cdots > b_k^+ \to -\infty$. If we let

$$D^{(k)} = (b_{k+1}^+, b_k^+],\ k \in \mathbf{N},$$

then

$$D^{(0)} + b + c \subseteq \mathbf{R}^+,$$

$$D^{(k)} + b + c = D^{(k-1)},\ k \ge 1,$$

and

$$\mathbf{R}^- = \bigcup_{k=0}^{\infty} D^{(k)}.$$

By methods similar to the proof of the Theorem 2.3, we may easily obtain the following result.

Theorem 2.4. Suppose $a = 1, b > 0$ and $c - b = 0 < c + b$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behavior can be summarized in the following table:

Table 3:

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$\in \mathbf{R}^+$	$\in \mathbf{R}^+$		$\to x_{-2}$	$\to x_{-1}$
$\in \mathbf{R}^-$	$\in \mathbf{R}^+$		$\to x_{-2}$	$\to +\infty$
$\in \mathbf{R}^+$	$\in \mathbf{R}^-$		$\to +\infty$	$\to x_{-1}$
$\in D^{(k)}$	$\in D^{(s)}$	$0 \le s < k$	$\to x_{2s}$	$\to +\infty$
$\in D^{(k)}$	$\in D^{(s)}$	$0 \le k \le s$	$\to +\infty$	$\to x_{2k-1}$

In the next result, we assume that $a = 1, b > 0$ and $c - b < 0 < c + b$. Then $b_0^- < b_1^- < \cdots < b_k^- \to +\infty$ and $b_0^+ > b_1^+ > \cdots > b_k^+ \to -\infty$. We may therefore use the same notations $C^{(k)}$ and $D^{(k)}$ in the previous two cases.

Theorem 2.5. Suppose $a = 1, b > 0$ and $c - b < 0 < c + b$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^- \cup C^{(k)}C^{(s)} \cup D^{(r)}D^{(t)}$ where $0 \le s < k$ and $0 \le r \le t$, then $\lim_n x_{2n} = +\infty$ and $\lim_n x_{2n+1} = -\infty$.

(ii) If $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^+ \cup C^{(k)}C^{(s)} \cup D^{(r)}D^{(t)}$ where $0 \le k \le s$ and $0 \le t < r$, then $\lim_n x_{2n} = -\infty$ and $\lim_n x_{2n+1} = +\infty$.

Proof. (i) Suppose $(x_{-2}, x_{-1}) \in C^{(k)}C^{(s)}$ where $0 \le s < k$. Then by (15),

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} - b + c \in C^{(k)} - b + c = C^{(k-1)}, \\ x_1 &= x_{-1} + bH(x_0) + c = x_{-1} - b + c \in C^{(s)} - b + c = C^{(s-1)}, \end{aligned}$$

and by induction, $(x_{2s}, x_{2s+1}) \in \mathbf{R}^+\mathbf{R}^-$. Suppose $(x_{-2}, x_{-1}) \in D^{(r)}D^{(t)}$ where $0 \le r \le t$. Then by (15), if $r = 0$, then

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} + b + c \in D^{(0)} + b + c = D^{(-1)} \subseteq \mathbf{R}^+, \\ x_1 &= x_{-1} + bH(x_0) + c = x_{-1} - b + c < x_{-1} \le 0, \end{aligned}$$

i.e., $(x_0, x_1) \in \mathbf{R}^+\mathbf{R}^-$; while if $r > 0$, then

$$\begin{aligned} x_0 &= x_{-2} + bH(x_{-1}) + c = x_{-2} + b + c \in D^{(r)} - b + c = D^{(r-1)}, \\ x_1 &= ax_{-1} + bH(x_0) + c = x_{-1} + b + c \in D^{(t)} - b + c = D^{(t-1)}, \end{aligned}$$

and by induction, $(x_{2r}, x_{2r+1}) \in \mathbf{R}^+\mathbf{R}^-$.

Therefore, we may suppose without loss of generality that $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. By (15) and induction, we may then see that $(x_{2n}, x_{2n+1}) \in \mathbf{R}^+\mathbf{R}^-$ for all $n \in \mathbf{N}$. Thus $x_{2n} = x_{2n-2} + b + c$ and $x_{2n+1} = x_{2n-1} - b + c$ for $n \in \mathbf{N}$. In view of (8) and (9), $\lim_{n\to\infty} x_{2n} = +\infty$ and $\lim_{n\to\infty} x_{2n+1} = -\infty$ as desired.

(ii) This case is similar to the case (i) and its proof is skipped.

The proof is complete.

3 The Case where $a = 1$ and $b < 0$

Under the assumption that $a = 1$ and $b < 0$, equation (15) is equivalent to

$$x_n = ax_{n-2} + \widetilde{b}\widetilde{H}(x_{n-1}) + c, \ n \in \mathbf{N},$$

where $\widetilde{b} = -b > 0$ and $\widetilde{H} = -H$. Hence this case is similar to (but not the same as) that where $b > 0$ and $a = 1$. Indeed, under the assumption that $b < 0$ and $a = 1$, the corresponding asymptotic behavior of (15) only depends on $c + b$ and $c - b$. Since $c + b < c - b$, we again have five cases: (i) $c - b < 0$, (ii) $c - b = 0$, (iii) $c + b = 0$, (iv) $c + b < 0 < c - b$, and (v) $c + b > 0$.

The ideas of the proofs of the corresponding asymptotic behaviors are also similar, and hence some of the proofs in this section will be skipped or sketched.

Theorem 3.1. Suppose $a = 1, b < 0$ and $c - b < 0$. Then every solution of (15) tends to $-\infty$.

Theorem 3.2. Suppose $a = 1, b < 0$ and $c + b > 0$. Then every solution of (15) tends to $+\infty$.

Theorem 3.3. Suppose $a = 1, b < 0$ and $c - b = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^+$, then $\lim_n x_{2n} = x_{-2}$ and $\lim_n x_{2n+1} = x_{-1}$.

(ii) If $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^- \cup \mathbf{R}^-\mathbf{R}^+ \cup \mathbf{R}^-\mathbf{R}^-$, then $\lim_n x_n = -\infty$.

Proof. (i) Suppose $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^+$. Then by (15), $x_0 = x_{-2} - b + c = x_{-2}$, $x_1 = x_{-1} - b + c = x_{-1}$, and by induction, we may easily see that $x_{2n} = x_{-2}$ and $x_{2n+1} = x_{-1}$ for $n \in \mathbf{N}$.

(ii) Suppose $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then $x_{-2} \in (b_k^+, b_{k+1}^+]$ for some $k \in \mathbf{N}$. By (15) and induction, $(x_{2k-1}, x_{2k}) \in \mathbf{R}^-\mathbf{R}^-$. Suppose $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^+$. Then $x_{-1} \in (b_k^+, b_{k+1}^+]$ for some $k \in \mathbf{N}$. By (15) and induction, $(x_{2k}, x_{2k+1}) \in \mathbf{R}^-\mathbf{R}^-$. Therefore, we may suppose without loss of generality that $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$. Then by (15) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^-\mathbf{R}^-$ for all $n \geq -2$. Thus $x_n = x_{n-2} + b + c$ for $n \in \mathbf{N}$. In view of (8) and (9), $\lim_{n\to\infty} x_n = -\infty$.

Similar to Theorem 3.3, we may show the Theorem 3.4 as follows.

Theorem 3.4. Suppose $a = 1, b < 0$ and $c + b = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$, then $\lim_n x_{2n} = x_{-2}$ and $\lim_n x_{2n+1} = x_{-1}$ for all $n \in \mathbf{N}$.

(ii) If $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^- \cup \mathbf{R}^-\mathbf{R}^+ \cup \mathbf{R}^+\mathbf{R}^+$, then $\lim_n x_n = +\infty$.

In the next result, we assume that $a = 1, b < 0$ and $c + b < 0 < c - b$. Then $b_0^- > b_1^- > \cdots > b_k^- \to -\infty$ and $b_0^+ < b_1^+ < \cdots < b_k^+ \to +\infty$. Let

$$\overline{C}^{(k)} = (b_{k+1}^-, b_k^-], \overline{D}^{(k)} = (b_k^+, b_{k+1}^+]$$

for $k \in \mathbf{N}$, then

$$\mathbf{R}^- = \bigcup_{k=0}^{\infty} \overline{C}^{(k)}, \mathbf{R}^+ = \bigcup_{k=0}^{\infty} \overline{D}^{(k)}.$$

Theorem 3.5. Suppose $a = 1, b < 0, c + b < 0 < c - b$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in \overline{C}^{(k)}\overline{D}^{(s)} \cup \overline{D}^{(r)}\overline{C}^{(t)} \cup \mathbf{R}^-\mathbf{R}^-$ where $0 \le s < k$ and $0 \le r \le t$, then $\lim_n x_n = -\infty$.

(ii) If $(x_{-2}, x_{-1}) \in \overline{C}^{(k)}\overline{D}^{(s)} \cup \overline{D}^{(r)}\overline{C}^{(t)} \cup \mathbf{R}^+\mathbf{R}^+$ where $0 \le k \le s$ and $0 \le t < r$, then $\lim_n x_n = +\infty$.

Proof. (i) Suppose$(x_{-2}, x_{-1}) \in \overline{C}^{(k)}\overline{D}^{(s)}$ where $0 \le s < k$. Then as in the proof of Theorem 2.5, by (15) and induction, $(x_{2s}, x_{2s+1}) \in \mathbf{R}^-\mathbf{R}^-$. Suppose $(x_{-2}, x_{-1}) \in \overline{D}^{(r)}\overline{C}^{(t)}$ where $0 \le r \le t$. Then by (15) and induction, $(x_{2r-1}, x_{2r}) \in \mathbf{R}^-\mathbf{R}^-$. Therefore, we may suppose without loss of generality that $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$. Then by (15) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^-\mathbf{R}^-$ for all $n \ge -2$. Thus $x_n = x_{n-2} + b + c$ for $n \in \mathbf{N}$. In view of (8) and (9), $\lim_{n\to\infty} x_n = -\infty$.

(ii) This case is similar to the previous case (i) and its proof is skipped. where $\widetilde{b} = -b > 0$ and $\widetilde{H} = -H$. Hence this case is similar to (but not the same as) that where $b > 0$ and $a = 1$. Indeed, under the assumption that $b < 0$ and $a = 1$, the corresponding asymptotic behavior of (15) only depends on $c + b$ and $c - b$. Since $c + b < c - b$, we again have five cases: (i) $c - b < 0$, (ii) $c - b = 0$, (iii) $c + b = 0$, (iv) $c + b < 0 < c - b$, and (v) $c + b > 0$.

The ideas of the proofs of the corresponding asymptotic behaviors are also similar, and hence some of the proofs in this section will be skipped or sketched.

4 The Case where $a \in (0, 1)$ and $b > 0$

Under the assumption that $a \in (0, 1)$, we have $\alpha_- = (c-b)/(1-a) < (c+b)/(1-a) = \alpha_+$. Thus we need to consider five cases (i) $0 < \alpha_-$, (ii) $0 = \alpha_-$, (iii) $\alpha_- < 0 < \alpha_+$, (iv) $0 = \alpha_+$, and (v) $0 > \alpha_+$.

Theorem 4.1. Suppose $a \in (0, 1), b > 0$ and $\alpha_+ < 0$. Then every solution of (15) tends to α_+.

Proof. Let $\{x_k\}_{k=-2}^{\infty}$ be a solution of (15). If $x_k > 0$ for all $k \geq -2$, then from (15), $x_k = ax_{k-2} - b + c$ for all $k \in \mathbf{N}$. One sees immediately from (6) and (7) that $\lim_n x_n = (c - b)/(1 - a) = \alpha_- < \alpha_+ < 0$, which is a contradiction. Thus there is $m \geq -2$ such that $x_m \in \mathbf{R}^-$. Then we may show as in the proof of Theorem 2.1 that there exists $m \geq -2$ such that $x_m, x_{m+1} \in \mathbf{R}^-$. Therefore, we may suppose without loss of generality that $x_{-2}, x_{-1} \in \mathbf{R}^-$. Then by (15) and induction, $x_n \in \mathbf{R}^-$ for all $n \geq -2$. Thus $x_n = ax_{n-2} + b + c$ for $n \in \mathbf{N}$. In view of (6) and (7), $\lim_{n\to\infty} x_n = \alpha_+$. The proof is complete.

Theorem 4.2. Suppose $a \in (0, 1), b > 0$ and $\alpha_- > 0$. Then every solution of (15) tends to α_-.

The proof is similar to that of Theorem 4.1 and hence is omitted.

In the next result, we assume that $a \in (0, 1), b > 0$ and $\alpha_+ = 0$. Then $0 = a_0^- < a_1^- < \cdots < a_k^- \to +\infty$. If we let

$$A^{(k)} = (a_k^-, a_{k+1}^-],\ k \in \mathbf{N},$$

and

$$A^{(-1)} = aA^{(0)} - b + c = (-b + c, 0],$$

then

$$A^{(k-1)} = aA^{(k)} - b + c,\ k \in \mathbf{N},$$

$$A^{(-1)} \subseteq \mathbf{R}^-,$$

and

$$\mathbf{R}^+ = \bigcup_{k=0}^{\infty} A^{(k)}.$$

Theorem 4.3. Suppose $a \in (0, 1), b > 0$ and $0 = \alpha_+$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$, then $\lim_n x_n = 0$.

(ii) If $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup \mathbf{R}^+\mathbf{R}^-$ where $0 \le s < k$, then $\lim_n x_{2n} = 0$ and $\lim_n x_{2n+1} = \alpha_-$.

(iii) If $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup \mathbf{R}^-\mathbf{R}^+$ where $0 \le k \le s$, then $\lim_n x_{2n} = \alpha_-$ and $\lim_n x_{2n+1} = 0$.

Proof. The proof of (i) is quite easy in view of the proofs of Theorems 2.3, 2.4 and 2.5, and hence skipped. To see (ii), suppose $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)}$ where $0 \le s < k$. Then by (15),

$$\begin{aligned} x_0 &= ax_{-2} + bH(x_{-1}) + c = ax_{-2} - b + c \in aA^{(k)} - b + c = A^{(k-1)}, \\ x_1 &= ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \in aA^{(s)} - b + c = A^{(s-1)}, \end{aligned}$$

and by induction, $(x_{2s}, x_{2s+1}) \in A^{(k-s-1)}A^{(-1)} \subseteq \mathbf{R}^+\mathbf{R}^-$. Therefore, we may suppose without loss of generality $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then by (15) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^+\mathbf{R}^-$ for all $n \ge -2$. Thus $x_{2n} = ax_{2n-2} + b + c$ and $x_{2n+1} = ax_{2n-1} - b + c$. In view of (6) and (7), $\lim_{n\to\infty} x_{2n} = 0$ and $\lim_{n\to\infty} x_{2n+1} = \alpha_-$ as desired. Finally, the proof of (iii) is similar to that of the case (ii) and hence omitted.

In the next result, we assume that $a \in (0,1), b > 0$ and $\alpha_- = 0$. Then $0 = a_0^+ > a_1^+ > \cdots > a_k^+ \to -\infty$ and $(-\infty, 0) = \bigcup_{k=0}^{\infty}[a_{k+1}^+, a_k^+)$.

Theorem 4.4. Suppose $a \in (0,1), b > 0$ and $\alpha_- = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behavior can be summarized in the following table:

Table 4:

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$\in \mathbf{R}^+$	$\in \mathbf{R}^+$		$\to 0$	$\to 0$
$\in (a_1^+, +\infty)$	$= 0$		$\to \alpha_+$	$\to 0$
$\in \mathbf{R}^-$	$\in \mathbf{R}^+$		$\to 0$	$\to \alpha_+$
$\in (-\infty, a_1^+]$	$= 0$		$\to 0$	$\to \alpha_+$
$\in [0, +\infty)$	$\in (-\infty, 0)$		$\to \alpha_+$	$\to 0$
$\in [a_{k+1}^+, a_k^+)$	$\in [a_{s+1}^+, a_s^+)$	$0 \le k < s$	$\to \alpha_+$	$\to 0$
$\in [a_{k+1}^+, a_k^+)$	$= a_{s+1}^+$	$0 \le k = s$	$\to \alpha_+$	$\to 0$
$= a_{k+1}^+$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s$	$\to 0$	$\to \alpha_+$
$\in (a_{k+1}^+, a_k^+)$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s$	$\to \alpha_+$	$\to 0$
$= a_{k+1}^+$	$\in [a_{s+1}^+, a_s^+)$	$0 \le k = s + 1$	$\to 0$	$\to \alpha_+$
$\in (a_{k+1}^+, a_k^+)$	$= a_{s+1}^+$	$0 \le k = s + 1$	$\to \alpha_+$	$\to 0$
$\in (a_{k+1}^+, a_k^+)$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s + 1$	$\to 0$	$\to \alpha_+$
$\in [a_{k+1}^+, a_k^+)$	$\in [a_{s+1}^+, a_s^+)$	$k > s + 1$	$\to 0$	$\to \alpha_+$

Again the proof of Theorem 4.4 is similar to those of Theorems 2.3, 2.4 and 2.5, and hence skipped.

In the next result, we assume that $a \in (0,1), b > 0$ and $\alpha_- < 0 < \alpha_+$. Then $0 = a_0^- < a_1^- < \cdots < a_k^- \to +\infty$ and $0 = a_0^+ > a_1^+ > \cdots > a_k^+ \to -\infty$. Therefore, if let $A^{(k)} = (a_k^-, a_{k+1}^-]$ and $B^{(k)} = (a_{k+1}^+, a_k^+]$ for $k \in \mathbf{N}$ and $A^{(-1)} = aA^{(0)} - b + c$ and $B^{(-1)} = aB^{(0)} + b + c$, then

$$aA^{(k)} - b + c = A^{(k-1)}, aB^{(k)} + b + c = B^{(k-1)}, k \in \mathbf{N},$$

$$A^{(-1)} \subseteq \mathbf{R}^-, B^{(-1)} \subseteq \mathbf{R}^+,$$

and

$$\mathbf{R}^+ = \bigcup_{k=0}^{\infty} A^{(k)},\ \mathbf{R}^- = \bigcup_{k=0}^{\infty} B^{(k)}.$$

Theorem 4.5. Suppose $a \in (0,1), b > 0$ and $\alpha_- < 0 < \alpha_+$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15).

(i) If $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup B^{(r)}B^{(t)} \cup \mathbf{R}^+\mathbf{R}^-$ where $0 \le s < k$ and $0 \le r \le t$, then $\lim_n x_{2n} = \alpha_+$ and $\lim_n x_{2n+1} = \alpha_-$.

(ii) If $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup B^{(r)}B^{(t)} \cup \mathbf{R}^-\mathbf{R}^+$ where $0 \le k \le s$ and $0 \le t < r$, then $\lim_n x_{2n} = \alpha_-$ and $\lim_n x_{2n+1} = \alpha_+$.

Proof. Suppose $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)}$ where $0 \le s < k$. Then by (15),

$$\begin{aligned} x_0 &= ax_{-2} + bH(x_{-1}) + c = ax_{-2} - b + c \in aA^{(k)} - b + c = A^{(k-1)}, \\ x_1 &= ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \in aA^{(s)} - b + c = A^{(s-1)}, \end{aligned}$$

and by induction, $(x_{2s}, x_{2s+1}) \in A^{(k-s-1)}A^{(-1)} \subseteq \mathbf{R}^+\mathbf{R}^-$. Suppose $(x_{-2}, x_{-1}) \in B^{(r)}B^{(t)}$ where $0 \le r \le t$. By (15), if $r = 0$, then

$$\begin{aligned} x_0 &= ax_{-2} + bH(x_{-1}) + c = ax_{-2} + b + c \in aB^{(0)} + b + c = B^{(-1)} \subseteq \mathbf{R}^+, \\ x_1 &= ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \le -b + c < 0, \end{aligned}$$

i.e., $(x_0, x_1) \in \mathbf{R}^+\mathbf{R}^-$; while if $r > 0$, then

$$\begin{aligned} x_0 &= ax_{-2} + bH(x_{-1}) + c = ax_{-2} + b + c \in aB^{(r)} + b + c = B^{(r-1)}, \\ x_1 &= ax_{-1} + bH(x_0) + c = ax_{-1} + b + c \in aB^{(t)} + b + c = B^{(t-1)}, \end{aligned}$$

and by induction,$(x_{2r}, x_{2r+1}) \in \mathbf{R}^+\mathbf{R}^-$. Therefore, we may suppose without loss of generality that $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then by (15) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^+\mathbf{R}^-$ for all $n \ge -2$. Thus $x_{2n} = ax_{2n-2} + b + c$ and $x_{2n+1} = ax_{2n-1} - b + c$ for $n \in \mathbf{N}$. In view of (6) and (7), $\lim_{n\to\infty} x_{2n} = \alpha_+$ and $\lim_{n\to\infty} x_{2n+1} = \alpha_-$ as desired.

The conclusion (ii) is similar to (i) and its proof is omitted.

5 The Case where $a > 1$ and $b < 0$

Under the assumption that $a > 1$ and $b < 0$, we have $\alpha_- = (c-b)/(1-a) < (c+b)/(1-a) = \alpha_+$. We therefore need to consider five cases: (i) $\alpha_+ < 0$, (ii) $0 = \alpha_+$,(iii) $\alpha_- < 0 < \alpha_+$, (iv) $0 = \alpha_-$, and (v) $0 < \alpha_-$. Again, the conclusions below are quite similar to those above and their proofs are skipped.

Suppose $a > 1, b < 0$ and $\alpha_+ < 0$. Then $0 = a_0^+ > a_1^+ > \cdots > a_k^+ \longrightarrow \alpha_+$, $\alpha_- = a^+_{\alpha_-,0} < a^+_{\alpha_-,1} < \cdots < a^+_{\alpha_-,k} \longrightarrow \alpha_+$, $\alpha_+ = a^-_{\alpha_+,0} > a^-_{\alpha_+,1} > \cdots > a^-_{\alpha_+,k} \longrightarrow \alpha_-$ and

$$(\alpha_+, 0) = \bigcup_{k=0}^{\infty}[a^+_{k+1}, a^+_k), (\alpha_-, \alpha_+) = \bigcup_{k=0}^{\infty}(a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1}] = \bigcup_{k=0}^{\infty}[a^-_{\alpha_+,k+1}, a^-_{\alpha_+,k}).$$

Theorem 5.1. Suppose $a > 1, b < 0$ and $\alpha_+ < 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behaviors can be summarized in the following tables 5 and 6.

Table 5 ($(x_{-2}, x_{-1}) \in \mathbf{R}^2 \backslash \{(\alpha_+, 0) \times (\alpha_-, \alpha_+) \cup (\alpha_-, \alpha_+) \times (\alpha_+, 0)\}$) :

x_{-2}	x_{-1}	x_{2n}	x_{2n+1}
$\in (-\infty, \alpha_+)$	$\in (-\infty, \alpha_+)$	$\to -\infty$	$\to -\infty$
$= \alpha_+$	$\in (-\infty, \alpha_+)$	$\to \alpha_+$	$\to -\infty$
$\in (-\infty, \alpha_+)$	$= \alpha_+$	$\to -\infty$	$\to \alpha_+$
$= \alpha_+$	$= \alpha_+$	$\to \alpha_+$	$\to \alpha_+$
$\in (\alpha_+, +\infty)$	$\in (\alpha_+, +\infty)$	$\to +\infty$	$\to +\infty$
$\in (\alpha_+, +\infty)$	$= \alpha_+$	$\to +\infty$	$\to +\infty$
$= \alpha_+$	$\in (\alpha_+, +\infty)$	$\to +\infty$	$\to +\infty$
$\in [0, +\infty)$	$\in (\alpha_-, \alpha_+)$	$\to +\infty$	$\to +\infty$
$\in (\alpha_-, \alpha_+)$	$\in \mathbf{R}^+$	$\to +\infty$	$\to +\infty$
$\in (a^+_{\alpha_-,1}, \alpha_+]$	$= 0$	$\to +\infty$	$\to +\infty$
$\in (\alpha_+, +\infty)$	$\in (-\infty, \alpha_-)$	$\to +\infty$	$\to -\infty$
$\in (\alpha_+, a_1^+]$	$= \alpha_-$	$\to +\infty$	$\to -\infty$
$\in (-\infty, \alpha_-)$	$\in (\alpha_+, +\infty)$	$\to -\infty$	$\to +\infty$
$= \alpha_-$	$\in (\alpha_+, 0]$	$\to -\infty$	$\to +\infty$
$\in (\alpha_-, a^+_{\alpha_-,1})$	$= 0$	$\to -\infty$	$\to +\infty$
$\in (a_1^+, +\infty)$	$= \alpha_-$	$\to +\infty$	$\to \alpha_-$
$= \alpha_-$	$\in \mathbf{R}^+$	$\to \alpha_-$	$\to +\infty$
$= a^+_{\alpha_-,1}$	$= 0$	$\to \alpha_-$	$\to +\infty$

Table 6 $((x_{-2}, x_{-1}) \in (\alpha_+, 0) \times (\alpha_-, \alpha_+) \cup (\alpha_-, \alpha_+) \times (\alpha_+, 0))$:

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$\in [a^+_{k+1}, a^+_k)$	$\in (a^+_{\alpha_-,s}, a^+_{\alpha_-,s+1}]$	$k < s$	$\to +\infty$	$\to +\infty$
$\in (a^+_{k+1}, a^+_k)$	$\in (a^+_{\alpha_-,s}, a^+_{\alpha_-,s+1}]$	$k = s$	$\to +\infty$	$\to +\infty$
$= a^+_{k+1}$	$\in (a^+_{\alpha_-,s}, a^+_{\alpha_-,s+1})$	$k = s$	$\to +\infty$	$\to -\infty$
$= a^+_{k+1}$	$= a^+_{\alpha_-,s+1}$	$k = s$	$\to +\infty$	$\to \alpha_-$
$\in [a^+_{k+1}, a^+_k)$	$\in (a^+_{\alpha_-,s}, a^+_{\alpha_-,s+1})$	$k = s+1$	$\to +\infty$	$\to -\infty$
$\in (a^+_{k+1}, a^+_k)$	$= a^+_{\alpha_-,s+1}$	$k = s+1$	$\to +\infty$	$\to \alpha_-$
$= a^+_{k+1}$	$= a^+_{\alpha_-,s+1}$	$k = s+1$	$\to +\infty$	$\to -\infty$
$\in [a^+_{k+1}, a^+_k)$	$\in (a^+_{\alpha_-,s}, a^+_{\alpha_-,s+1}]$	$k > s+1$	$\to +\infty$	$\to -\infty$
$\in (a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1}]$	$\in [a^+_{s+1}, a^+_s)$	$k < s$	$\to -\infty$	$\to +\infty$
$\in (a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1})$	$\in [a^+_{s+1}, a^+_s)$	$k = s$	$\to -\infty$	$\to +\infty$
$= a^+_{\alpha_-,k+1}$	$\in (a^+_{s+1}, a^+_s)$	$k = s$	$\to \alpha_-$	$\to +\infty$
$= a^+_{\alpha_-,k+1}$	$= a^+_{s+1}$	$k = s$	$\to -\infty$	$\to +\infty$
$\in (a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1}]$	$\in (a^+_{s+1}, a^+_s)$	$k = s+1$	$\to +\infty$	$\to +\infty$
$\in (a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1})$	$= a^+_{s+1}$	$k = s+1$	$\to -\infty$	$\to +\infty$
$= a^+_{\alpha_-,k+1}$	$= a^+_{s+1}$	$k = s+1$	$\to \alpha_-$	$\to +\infty$
$\in (a^+_{\alpha_-,k}, a^+_{\alpha_-,k+1}]$	$\in [a^+_{s+1}, a^+_s)$	$k > s+1$	$\to +\infty$	$\to +\infty$

Table 7:

x_{-2}	x_{-1}	x_{2n}	x_{2n+1}
$\in (-\infty, 0)$	$\in (-\infty, 0)$	$\to -\infty$	$\to -\infty$
$\in (-\infty, 0)$	$= 0$	$\to -\infty$	$\to 0$
$= 0$	$\in (-\infty, 0)$	$\to 0$	$\to -\infty$
$= 0$	$= 0$	$\to 0$	$\to 0$
$\in \mathbf{R}^+$	$\in R^+$	$\to +\infty$	$\to +\infty$
$\in \mathbf{R}^+$	$= 0$	$\to +\infty$	$\to +\infty$
$= 0$	$\in \mathbf{R}^+$	$\to +\infty$	$\to +\infty$
$\in \mathbf{R}^+$	$\in (\alpha_-, 0)$	$\to +\infty$	$\to +\infty$
$\in (\alpha_-, 0)$	$\in \mathbf{R}^+$	$\to +\infty$	$\to +\infty$
$\in \mathbf{R}^+$	$\in (-\infty, \alpha_-)$	$\to +\infty$	$\to -\infty$
$\in \mathbf{R}^+$	$= \alpha_-$	$\to +\infty$	$\to \alpha_-$
$\in (-\infty, \alpha_-)$	$\in \mathbf{R}^+$	$\to -\infty$	$\to +\infty$
$= \alpha_-$	$\in \mathbf{R}^+$	$\to \alpha_-$	$\to +\infty$

In the next result, we assume that $a > 1, b < 0$ and $\alpha_- < \alpha_+ = 0$. Then $a^-_{\alpha_+,0} > a^-_{\alpha_+,1} > \cdots > a^-_{\alpha_+,k} \to \alpha_-$ and

$$(\alpha_-, 0) = \bigcup_{k=0}^{\infty} [a^-_{\alpha_+,k+1}, a^-_{\alpha_+,k}).$$

Theorem 5.2. Suppose $a > 1, b < 0$ and $\alpha_+ = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behaviors can be summarized in the above table 7.

In the next result, we assume that $a > 1, b < 0$ and $\alpha_- < 0 < \alpha_+$. Then $0 = a_0^+ < a_1^+ < \cdots < a_k^+ \to \alpha_+$ and $0 = a_0^- > a_1^- > \cdots > a_k^- \to \alpha_-$. Let

$$G^{(k)} = (a_k^+, a_{k+1}^+],\ H^{(k)} = [a_{k+1}^-, a_k^-)$$

for $k \in \mathbf{N}$, then

$$(0, \alpha_+) = \bigcup_{k=0}^{\infty} G^{(k)}, (\alpha_-, 0) = \bigcup_{k=0}^{\infty} H^{(k)}.$$

Theorem 5.3. Suppose $a > 1, b < 0$ and $\alpha_- < 0 < \alpha_+$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behaviors can be summarized in the following tables 8 and 9.

Table 8 ($(x_{-2}, x_{-1}) \in [\alpha_+, +\infty) \times (-\infty, \alpha_-] \cup (-\infty, \alpha_-] \times [\alpha_+, +\infty)$):

x_{-2}	x_{-1}	x_{2n}	x_{2n+1}
$(\alpha_+, +\infty)$	$(-\infty, \alpha_-)$	$\to +\infty$	$\to -\infty$
α_+	$(-\infty, \alpha_-)$	$\to \alpha_+$	$\to -\infty$
$(\alpha_+, +\infty)$	α_-	$\to +\infty$	$\to \alpha_-$
α_+	α_-	$\to \alpha_+$	$\to \alpha_-$
$(-\infty, \alpha_-)$	$(\alpha_+, +\infty)$	$\to -\infty$	$\to +\infty$
$(-\infty, \alpha_-)$	α_+	$\to -\infty$	$\to \alpha_+$
α_-	$(\alpha_+, +\infty)$	$\to \alpha_-$	$\to +\infty$
α_-	α_+	$\to \alpha_-$	$\to \alpha_+$

Table 9 ($(x_{-2}, x_{-1}) \in \mathbf{R}^2 \backslash \{[\alpha_+, +\infty) \times (-\infty, \alpha_-] \cup (-\infty, \alpha_-] \times [\alpha_+, +\infty)\}$):

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$\mathbf{R}^-$	$\mathbf{R}^-$		$\to -\infty$	$\to -\infty$
$(-\infty, \alpha_-]$	$(0, \alpha_+)$		$\to -\infty$	$\to -\infty$
$(0, \alpha_+)$	$(-\infty, \alpha_-]$		$\to -\infty$	$\to -\infty$
$(0, a_1^+]$	0		$\to -\infty$	$\to -\infty$
$G^{(k)}$	$H^{(s)}$	$k \leq s$	$\to -\infty$	$\to -\infty$
$H^{(r)}$	$G^{(t)}$	$t < r$	$\to -\infty$	$\to -\infty$
$\mathbf{R}^+$	$\mathbf{R}^+$		$\to +\infty$	$\to +\infty$
$[\alpha_+, +\infty)$	$(\alpha_-, 0)$		$\to +\infty$	$\to +\infty$
$(\alpha_-, 0)$	$[\alpha_+, +\infty)$		$\to +\infty$	$\to +\infty$
0	$\mathbf{R}^+$		$\to +\infty$	$\to +\infty$
$(a_1^+, +\infty)$	0		$\to +\infty$	$\to +\infty$
$G^{(k)}$	$H^{(s)}$	$s < k$	$\to +\infty$	$\to +\infty$
$H^{(r)}$	$G^{(t)}$	$r \leq t$	$\to +\infty$	$\to +\infty$

Table 10:

x_{-2}	x_{-1}	x_{2n}	x_{2n+1}
$\in \mathbf{R}^+$	$\in \mathbf{R}^+$	$\to +\infty$	$\to +\infty$
$\in \mathbf{R}^-$	$\in \mathbf{R}^-$	$\to -\infty$	$\to -\infty$
$\in (0, \alpha_+)$	$\in \mathbf{R}^-$	$\to -\infty$	$\to -\infty$
$\in \mathbf{R}^-$	$\in (0, \alpha_+)$	$\to -\infty$	$\to -\infty$
$\in (\alpha_+, +\infty)$	$\in (-\infty, 0)$	$\to +\infty$	$\to -\infty$
$\in (-\infty, 0)$	$\in (\alpha_+, +\infty)$	$\to -\infty$	$\to +\infty$
$= \alpha_+$	$\in (-\infty, 0)$	$\to \alpha_+$	$\to -\infty$
$\in (-\infty, 0)$	$= \alpha_+$	$\to -\infty$	$\to \alpha_+$
$\in (\alpha_+, +\infty)$	$= 0$	$\to +\infty$	$\to 0$
$= 0$	$\in (\alpha_+, +\infty)$	$\to 0$	$\to +\infty$
$= 0$	$= \alpha_+$	$\to 0$	$\to \alpha_+$
$= \alpha_+$	$= 0$	$\to \alpha_+$	$\to 0$

In the next result, we assume that $a > 1, b < 0$ and $\alpha_- = 0$. Then $0 = a_0^+ < a_1^+ < \cdots < a_k^+ \to \alpha_+$ and

$$(0, \alpha_+) = \bigcup_{k=0}^{\infty} (a_k^+, a_{k+1}^+].$$

Theorem 5.4. Suppose $a > 1, b < 0$ and $0 = \alpha_-$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then its limiting behaviors can be summarized in the above table 10.

In the next result, we assume that $a > 1, b < 0$ and $0 < \alpha_-$. Then $0 = a_0^- < a_1^- < \cdots < a_k^- \to \alpha_-$, $\alpha_- = a_{\alpha-,0}^+ < a_{\alpha_-,1}^+ < \cdots < a_{\alpha_-,k}^+ \longrightarrow \alpha_+$, $\alpha_+ = a_{\alpha+,0}^- > a_{\alpha_+,1}^- > \cdots > a_{\alpha_+,k}^- \longrightarrow \alpha_-$ and

$$(0, \alpha_-) = \bigcup_{k=0}^{\infty} (a_k^-, a_{k+1}^-],\ (\alpha_-, \alpha_+) = \bigcup_{k=0}^{\infty} (a_{\alpha_-,k}^+, a_{\alpha_-,k+1}^+] = \bigcup_{k=0}^{\infty} [a_{\alpha_+,k+1}^-, a_{\alpha_+,k}^-).$$

We also need to break up R^2 into six different parts: (a) $\Gamma_1 = [\alpha_-, \infty)^2$, (b) $\Gamma_2 = [\alpha_+, \infty) \times (-\infty, \alpha_-)$, (c) $\Gamma_3 = (-\infty, \alpha_-) \times [\alpha_+, \infty)$, (d) $\Gamma_4 = (0, \alpha_-) \times (\alpha_-, \alpha_+)$, (e) $\Gamma_5 = (\alpha_-, \alpha_+) \times (0, \alpha_-)$, and (f) $\Gamma_6 = \mathbf{R}^2 \backslash \bigcup_{i=1}^{5} \Gamma_i$.

Theorem 5.5. Suppose $a > 1, b < 0$ where $0 < \alpha_-$. Let $\{x_n\}_{n=-2}^{\infty}$ be any solution of (15). Then $\lim_n x_n = -\infty$ if $(x_{-2}, x_{-1}) \in \Gamma_6$, else its limiting behaviors can be summarized in the following table:

Table 11 ($(x_{-2}, x_{-1}) \in \bigcup_{i=1}^{5} \Gamma_i$):

x_{-2}	x_{-1}	condition	x_{2n}	x_{2n+1}
$(\alpha_-, +\infty)$	$(\alpha_-, +\infty)$		$\to +\infty$	$\to +\infty$
α_-	$(\alpha_-, +\infty)$		$\to \alpha_-$	$\to +\infty$
$(\alpha_-, +\infty)$	α_-		$\to +\infty$	$\to \alpha_-$
α_-	α_-		$\to \alpha_-$	$\to \alpha_-$
$(\alpha_+, +\infty)$	$(-\infty, \alpha_-)$		$\to +\infty$	$\to -\infty$
α_+	$(0, \alpha_-)$		$\to +\infty$	$\to -\infty$
α_+	R^-		$\to \alpha_+$	$\to -\infty$
$(-\infty, \alpha_-)$	$(\alpha_+, +\infty)$		$\to -\infty$	$\to +\infty$
(a_1^-, α_-)	α_+		$\to -\infty$	$\to +\infty$
$(-\infty, a_1^-]$	α_+		$\to -\infty$	$\to \alpha_+$
$(a_k^-, a_{k+1}^-]$	$[a_{\alpha_+,s+1}^-, a_{\alpha_+,s}^-)$	$0 \le k \le s$	$\to -\infty$	$-\infty$
$(a_k^-, a_{k+1}^-]$	$a_{\alpha_+,s+1}^-$	$k = s+1$	$\to -\infty$	$\to \alpha_+$
$(a_k^-, a_{k+1}^-]$	$(a_{\alpha_+,s+1}^-, a_{\alpha_+,s}^-)$	$k = s+1$	$\to -\infty$	$\to +\infty$
$(a_k^-, a_{k+1}^-]$	$[a_{\alpha_+,s+1}^-, a_{\alpha_+,s}^-)$	$k > s+1$	$\to -\infty$	$\to +\infty$
$[a_{\alpha_+,k+1}^-, a_{\alpha_+,k}^-)$	$(a_s^-, a_{s+1}^-]$	$0 \le s < k$	$\to -\infty$	$\to -\infty$
$a_{\alpha_+,k+1}^-$	$(a_s^-, a_{s+1}^-]$	$k = s$	$\to \alpha_+$	$\to -\infty$
$(a_{\alpha_+,k+1}^-, a_{\alpha_+,k}^-)$	$(a_s^-, a_{s+1}^-]$	$k = s$	$\to +\infty$	$\to -\infty$
$[a_{\alpha_+,k+1}^-, a_{\alpha_+,k}^-)$	$(a_s^-, a_{s+1}^-]$	$k < s$	$\to +\infty$	$\to -\infty$

6 Concluding Remarks

Since we have derived the exact relations between the initial pair (x_{-2}, x_{-1}) with the limiting behaviors of the solution $\{x_k\}_{k=-2}^{\infty}$ of (15) originated from it, we may make some interesting observations. For instance,

- in case $a = 1$ and $b > 0$, a solution $\{x_k\}_{k=-2}^{\infty}$ of (15) converges if, and only if, $c + b = 0$ and $x_{-2} = x_{-1} \in \mathbf{R}^-$, or, $c - b = 0$ and $x_{-2} = x_{-1} \in \mathbf{R}^+$;

- in case $a = 1$ and $b < 0$, a solution $\{x_k\}_{k=-2}^{\infty}$ of (15) converges if, and only if, $c - b = 0$ and $x_{-2} = x_{-1} \in \mathbf{R}^+$, or, $c + b = 0$ and $x_{-2} = x_{-1} \in \mathbf{R}^-$;

- in case $a \in (0, 1)$ and $b > 0$, a solution $\{x_k\}_{k=-2}^{\infty}$ of (15) converges if, and only if, (i) $\alpha_+ < 0$, (ii) $\alpha_- > 0$, (iii) $\alpha_+ = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^-$; or, (iv) $\alpha_- = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^+$.

- in case $a > 1$ and $b < 0$, a solution $\{x_k\}_{k=-2}^{\infty}$ of (15) converges if, and only if, (i) $\alpha_+ \le 0$ and $x_{-2} = x_{-1} = \alpha_+$; (ii) $\alpha_- > 0$ and $x_{-2} = x_{-1} = \alpha_-$.

We may also make assertions on the limiting behaviors of subsequences $\{x_{2k}\}_{k=-1}^{\infty}$ and $\{x_{2k+1}\}_{k=-1}^{\infty}$ of solutions $\{x_k\}_{k=-2}^{\infty}$ of (15). These and others can be made by going through the previous results one by one, and are not listed here since they do not offer any new theoretical information.

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