Annual Review of Chaos Theory, Bifurcations and Dynamical Systems
Vol. 8, (2019) 29-37, www.arctbds.com.

Golden Ratios, Lucas Sequences and the Quadratic Family

Arturo Ortiz-Tapia, Universidad Abierta y a Distancia de México
e-mail: aortiztapia2013@gmail.com

Abstract: It is conjectured that there is a converging sequence of some generalized Fibonacci ratios, given the difference between consecutive ratios, such as the golden ratio φ^1 and the next golden ratio φ^2. Moreover, the graphic depiction of those ratios shows some overlap with the quadratic family, and some numerical evidence suggests that all those ratios in the finite set obtained, belong to at least one quadratic family.

Keywords: Quadratic family, golden ratio, generalized Lucas sequence.

Manuscript accepted July 12, 2019.

1 Introduction

1.1 The Golden Ratio and Beauty

What is *beauty*? When can something be called *beautiful*? Beauty has been defined in many different ways throughout cultures [4]. An ancient attempt to define beauty systematically, was through the golden ratio. Golden ratio is the number $\varphi \approx 1.6180339887\cdots$, and its inverse $1/\varphi \approx 0.6180339887\cdots$ which is often associated with it. The number φ can be defined in several ways, one of them is through a recurrent process involving the Fibonacci numbers [6]

$$\varphi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \tag{1}$$

where

$$F_n = F_{n-1} + F_{n-2} \tag{2}$$

with initial values $F_0 = 1,\ F_1 = 1$.

1.2 Lucas Numbers

Actually, one can start with other initial values, say $F_0 = 1$, $F_1 = 3$ and still converge to the same value of φ (or its inverse) when the proportion between predecessor and successor is taken as $n \to \infty$. The sequence of numbers is known as Lucas sequence, and the component numbers, Lucas numbers [8]. In fact, it shouldn't be hard to prove that all the initial values imposed in (Eq. 2) do is to shift individual members of the Lucas sequence, but again they will converge to φ.

2 Other φ's

2.1 Positive φ's

So, how can one obtain a convergence to a value other than φ ? It turns out that one has to add more members to the recurrence in (Eq. 2), for example [5, 7]

$$L_n = L_{n-1} + L_{n-2} + \cdots + L_{n-k}, k \in +\mathbb{Z} \tag{3}$$

without loss of generality, the initial values to these Lucas sequences can be stated as

$$L_0 = 1,\ L_1 = 1, \cdots L_{k-1} = 1 \tag{4}$$

and let be defined φ^k be the value to which convergences (Eq.5)

$$\varphi^k = \lim_{n\to\infty} \frac{L_{n+1}^k}{L_n^k} \tag{5}$$

or their inverse

$$1/\varphi^k = \lim_{n\to\infty} \frac{L_n^k}{L_{n+1}^k} \tag{6}$$

For convenience, from now on inverses will be used, and for simplicity they will be called φ^k. For the purposes of this paper, the Lucas numbers were calculated using the Wolfram Mathematica command `RecurrenceTable` (with $n = 100$, i.e., 100 terms were taken), and then each φ^k is calculated as in Eq. 6. Although it is of course possible to define an indefinite amount of fractions like in (Eq. 6), it appears that there are only nine that are real, positive, and forming a totally-ordered set (Table 1). The last statement implies that $\varphi^k - \varphi^{k+1} > 0$. Exceptions to this rule begin, of course, when $k > 10$.

2.2 Towards Convergence of φ's

As interesting as the set listed on Table 1 might be, it is important to revise if further on there are more positive differences or not, and to look either for a forming pattern or convergence. So, calculating more Lucas sequences, it is obtained (Eq. 7)

k	φ^k
1	$0.618034\cdots$
2	$0.543689\cdots$
3	$0.51879\cdots$
4	$0.50866\cdots$
5	$0.504138\cdots$
6	$0.502017\cdots$
7	$0.500994\cdots$
8	$0.500493\cdots$
9	$0.500245\cdots$

Table 1: List of φ^k, $k \in \{1 \cdots 9\}$, $k \in +\mathbb{Z}$. Notice that φ^1 is the golden ratio.

$$\begin{pmatrix} k & \varphi^k \\ 1 & 0.618034 \\ 2 & 0.543689 \\ 3 & 0.51879 \\ 4 & 0.50866 \\ 5 & 0.504138 \\ 6 & 0.502017 \\ 7 & 0.500994 \\ 8 & 0.500493 \\ 9 & 0.500245 \\ 10 & 0.500122 \\ 11 & 0.500061 \\ 12 & 0.500031 \\ 13 & 0.500015 \\ 14 & 0.500008 \\ 15 & 0.500004 \\ 16 & 0.500002 \\ 17 & 0.500001 \\ 18 & 0.5 \\ 19 & 0.5 \\ 20 & 0.5 \\ 21 & 0.5 \\ 22 & 0.5 \\ 23 & 0.5 \\ 24 & 0.5 \\ 25 & 0.5 \\ 26 & 0.5 \\ 27 & 0.5 \\ 28 & 0.5 \\ 29 & 0.5 \\ 30 & 0.5 \end{pmatrix} \tag{7}$$

Apparently, numerical convergence has been attained at $\varphi^{18} = 0.5$. The sequence of differences is (Eq. 8)

$$\begin{pmatrix} k & \varphi^{k+1} - \varphi^k \\ 1 & 0.074345 \\ 2 & 0.0248989 \\ 3 & 0.0101297 \\ 4 & 0.00452213 \\ 5 & 0.0021212 \\ 6 & 0.00102288 \\ 7 & 0.00050106 \\ 8 & 0.000247656 \\ 9 & 0.000123033 \\ 10 & 0.0000612972 \\ 11 & 0.0000305886 \\ 12 & 0.0000152779 \\ 13 & 7.634520830523961^{-6} \\ 14 & 3.816065719419726^{-6} \\ 15 & 1.9077125118505123^{-6} \\ 16 & 9.537707341689128^{-7} \\ 17 & 4.768626257201092^{-7} \\ 18 & 2.384252867360104^{-7} \\ 19 & 1.1921105180778824^{-7} \\ 20 & 5.960510662816887^{-8} \\ 21 & 2.980244317996039^{-8} \\ 22 & 1.4901192724181556^{-8} \\ 23 & 7.45058881257421^{-9} \\ 24 & 3.725292407885661^{-9} \\ 25 & 1.8626457043424693^{-9} \\ 26 & 9.313227966600834^{-10} \\ 27 & 4.656612873077393^{-10} \\ 28 & 2.3283064365386963^{-10} \\ 29 & 1.1641532182693481^{-10} \end{pmatrix} \tag{8}$$

which is monotonic and bounded, and can therefore be taken as a converging sequence.

3 Matrix Form of Lucas Sequences and Its Eigenvalues

Eq. 2 can be represented in matrix form [7], first transforming the recurrence relation into a linear system by adding one more equation

$$F_{n-1} = F_{n-1} + (0) \cdot F_{n-2} \tag{9}$$

Thus

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} + 0 \end{aligned} \tag{10}$$

and then in matrix notation

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \tag{11}$$

Now, the eigenvalues of the matrix in Eq. 11 are

$$\lambda_{1,1} = \frac{1}{2}\left(1+\sqrt{5}\right) = 1/\varphi \tag{12}$$

and

$$\lambda_{1,2} = \frac{1}{2}\left(1-\sqrt{5}\right) \tag{13}$$

It is possible to construct matrices for all the Lucas sequences until convergence of the φ^k. The elements of the general matrix A_k for obtaining the rest of the eigenvalues for the corresponding φ^k, $k \geq 2$, are defined as follows:

$$\begin{aligned} a_{1j} &= 1,\ 1 \leq j \leq k+1,\ 2 \leq k \leq 19 \\ a_{2,1} &= 1 \\ a_{j+2,j+1} &= 1,\ 1 \leq j \leq k-1 \end{aligned} \tag{14}$$

It is known from linear algebra [1] that a $n \times n$ matrix will have n eigenvalues, so once the eigenvalues $\lambda_{k,n}$ are obtained, it is possible to obtain $\varphi_{k,n} = 1/\lambda_{k,n}$, just as in Eq. 12.

4 The $\varphi_{k,n}$ and the Quadratic Family

Once it is obtained the entire set of $\varphi_{k,n}$, it is possible to plot them in the complex plane as it can be seen in Fig. 1

Intuitively, Fig. 1 seems to have some resemblance to the Mandelbrot set, symbolized as $\mathcal{M}$. Here $\mathcal{M}$ is the set of all complex numbers c, for which the sequence $z_n = z_{n-1}^2 + c$ does not diverge to infinity when starting with $z_0 = 0$ [3, 2]. The resemblance is both in shape and approximate scale, as can be seen in Fig. (2) where the $\varphi_{k,n}$ are shown together with the Mandelbrot set.

Using the Wolfram Mathematica command `MandelbrotSetMemberQ`, it was tested whether $\varphi^{k,n} \in \mathcal{M}$. It turns out that 54 out of the 155 $\varphi^{k,n}$ *are* in $\mathcal{M}$; this suggests that for each of the characteristic polynomials of the A_k matrices, there are some functions $z_n = z_{n-1}^k + c$ which have overlapping values with some members of the quadratic family. Indeed, if one now turns its attention to the Julia set $\mathcal{J}$, where the set $\mathcal{J}$ of a function $f(z)$ is the closure of the set of all repelling fixed points of $f(z)$ [2], or phrased in another way, it is the points in the boundary of those points that do not escape to infinity under iteration [3]. The set of $\varphi^{k,n}$ is now plotted with the Julia set for comparison, as in Fig. 3

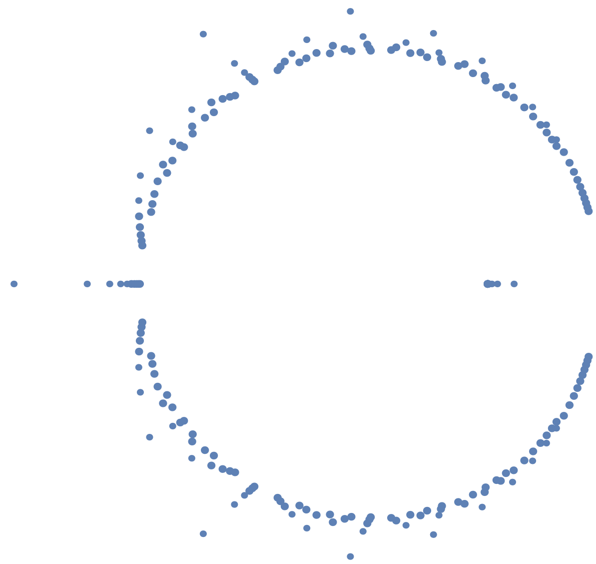

Figure 1: Plot of all $\varphi_{k,n}$.

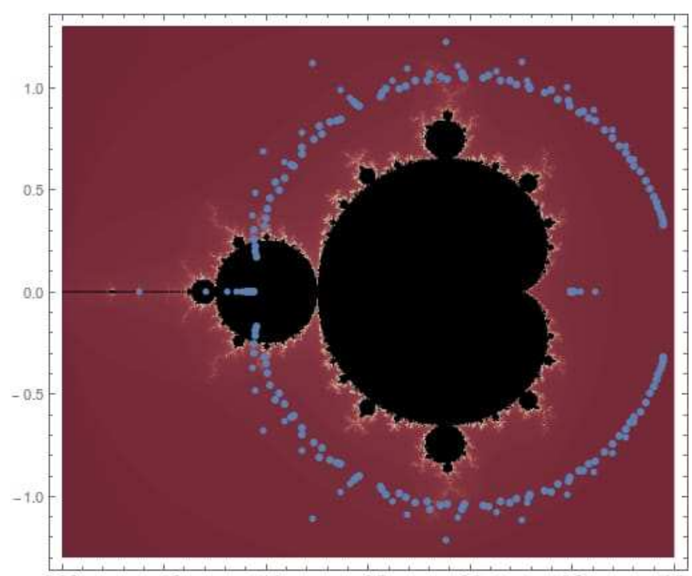

Figure 2: The complete set of $\varphi^{k,n}$ plotted together with $\mathcal{M}$.

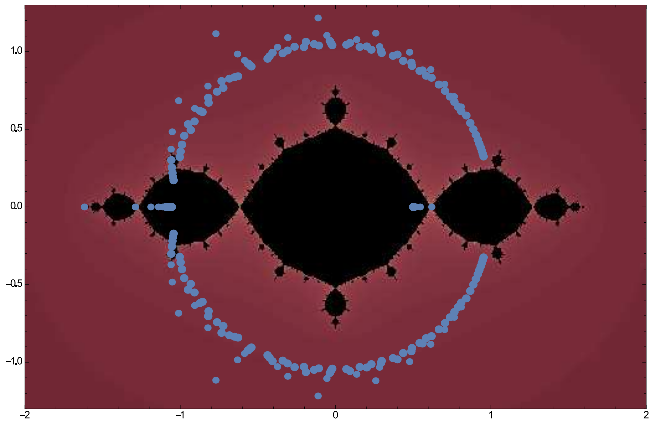

Figure 3: The complete set of $\varphi^{k,n}$ plotted together with $\mathcal{J}$.

Using the Wolfram Mathematica command `JuliaSetIterationCount[-1,` $\varphi^{k,n}$`]` (for the function $f(z) = z^2 + c,\ c = -1$), and with the criterion that membership to $\mathcal{J}$ is stablished when the number of iterations is $1000 + 1$, it is found out that only $\varphi_{1,1} = 0.618\cdots$ is in $\mathcal{J}$, for this $f(z)$. A depiction of this overlap is shown in Fig. 4. The procedure was repeated with `JuliaSetIterationCount[z2̂ - 2,` $\varphi^{k,n}$`]` (for the function $f(z) = z^2 + c,\ c = -2$), exhibiting that more members of the set of eigenvalues of the generalized Fibonacci matrices from the convergent sequence discussed in the subsection 2.2 belong to this Julia set, and this in turn seems to suggest that every member of that set belongs to at least one Julia set $f(z, c)$, for some c, and some polynomial functions defining the Julia set.

```
maxiterations = {};
Do[
  AppendTo[maxiterations,
   First[JuliaSetIterationCount[z^2 - 2, z, goldensetComplex[[k]]]]];
  If[numIt == 1001, inJulia += inJulia],
  {k, 1, Length[goldensetComplex]}];
Print[maxiterations]

{1001,1001,1001,1001,1001,1001,1001,1001,1001,3,3,2,2,2,2,2,2,2,2,2,
2,2,2,2,2,2,2,3,3,2,2,1001,3,3,2,2,2,2,1001,2,2,3,3,2,2,1001,2,2,2,2,
1001,2,2,1001,1001,3,3,2,2,2,2,2,2,4,4,1001,3,3,2,2,2,2,2,2,3,3,1001,
1001,3,3,2,2,2,2,2,2,2,2,4,4,1001,3,3,2,2,2,2,2,2,2,2,3,3,1001,1001,3,
3,2,2,2,2,2,2,2,2,2,3,3,4,4,1001,3,3,2,2,2,2,2,2,2,2,2,2,3,3,1001,1001,
3,3,2,2,2,2,2,2,2,2,2,2,3,3,4,4,1001,3,3,3,3,2,2,2,2,2,2,2,2,3,3,3,3,
1001,1001,3,3,3,3,2,2,2,2,2,2,2,2,2,2,3,3,4,4,1001,3,3,3,3,2,2,2,2,2,
2,2,2,2,2,3,3,3,3,1001}
```

Code that obtains the maximum number of iterations for each possible member of the Julia Set, thus exhibiting which eigenvalue is a member of the Julia set.

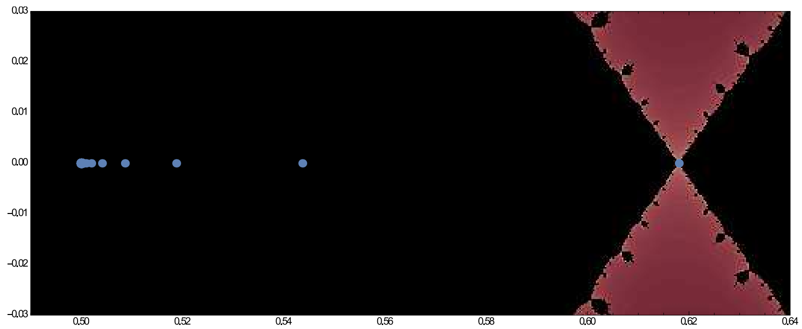

Figure 4: Zoom of $\mathcal{J}$ in the region where $\varphi_{1,1} \in \mathcal{J}$.

5 Conclusions

It was shown the existence of a set of golden ratios φ_k obtained by convergence of Lucas sequences. The Lucas sequences in turn were represented in matrix form to obtain $\varphi_{k,n}$ eigenvalues, where the graphic depiction and numerical evidence seem to suggest that every member of that set belongs to at least one member of the quadratic family, namely, the Julia set.

Acknowledgement: My deepest gratitude to my personal, inner muse, to whom I owe all the inspiration for this work. My sincere thanks to professors María del Carmen Lozano Arizmendi, Hugo H. Corrales and Oscar Adolfo Sanchez Valenzuela for suggesting extending the search of φ_k until convergence.

References

[1] H. Anton, *Elementary linear algebra*, John Wiley & Sons, 2010.

[2] R. Devaney, *An introduction to chaotic dynamical systems*, Westview press, 2008.

[3] R. L. Devaney, *Chaos, fractals, and dynamics: computer experiments in mathematics*, Addison-Wesley Longman Publishing Co., Inc., 1990.

[4] U. Eco and A. McEwen, *History of beauty*, 2005.

[5] S. Falcon, *On the k-lucas numbers*, Int. J. Contemp. Math. Sciences, 6 (2011), pp. 1039–1050.

[6] E. Lucas, *Théorie des nombres*, vol. 1, Gauthier-Villars, 1891.

[7] E. Miles, *Generalized Fibonacci numbers and associated matrices*, The American Mathematical Monthly, 67 (1960), pp. 745–752.

[8] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An introduction to the theory of numbers*, John Wiley & Sons, 1991.